15.4.5 Theorems of Green, Stokes and Gauss: An Introduction

The final sections of this text deal with the last three fundamental theorems of calculus, the theorems of Green, Stokes and Gauss. Each of these theorems extends the ideas of our earlier fundamental theorem of calculus to situations for vector-valued functions, and each has important applications to fields of physics and even to Maxwell's equations for magnetism and electricity. These theorems are technically sophisticated and difficult to prove, but the main ideas behind them are remarkably geometric and straightforward. The goal of this section is to approach these theorems geometrically and to illustrate why the ideas behind them are "easy and natural." The theorems will be clearly and precisely presented in the following sections and partial proofs will be given; the presentation in this section can help you understand what the theorems are saying and perhaps help you to remember them.

Introduction to Green's Theorem

Calculus deals with infinite collections of points, but sometimes a finite situation can give insight into the infinite.

In the following, R is a simple, simply-connected region consisting of a finite collection of cells. The boundary of R is a simple closed curve C (C consists of only the exterior edges of R). (Fig. 1)

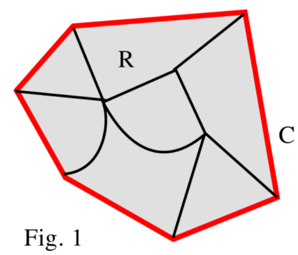


Fig. 1

Version 1 Green's Theorem: Divergence and Flux

Suppose water flows through the region R. Let's attach in-out flows to the edges of each cell. If we define the divergence of a cell to be the net outward flow of the cell then we can calculate the net outward flow of the collection of cells along the boundary of the collection – let's call this the flux across C.

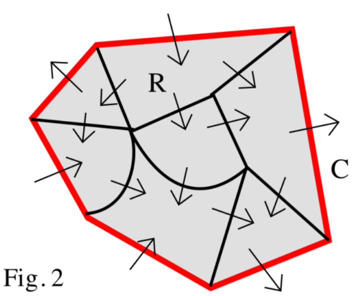


Fig. 2

We can calculate this flux in Fig. 2 in two ways. One way is to go around the boundary and add up the outward flows (counted as positive) and the inward flows (counted as negative). But if we add up the divergences for each cell in Fig. 2 we get the same net outward flow for the collection, the flux across C. This will always be true since for each inside boundary between cells, the outward flow from one cell becomes the inward flow into the next cell (Fig. 3) so the sum of those two flows will be zero, and that is the case for every shared edge inside the collection. Then the sum of all of the individual cell divergences is equal to just the sum of the flows on the outside edges (Fig. 4). This can be stated as

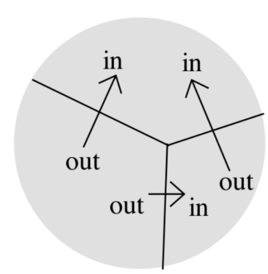


Fig. 3

In words: The sum of all of the individual cell divergences is equal to the sum of the flows on the outside edges (Fig. 4).

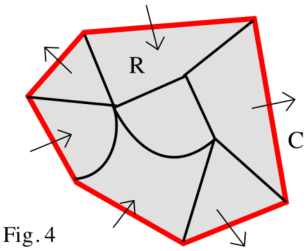


Fig. 4

Finite version (divergence-flux form Green's Theorem):

$$\sum_{R} \{\text{divergence of cell}\} = \sum_{C} \{\text{flow across each outside edge}\} = \{\text{flux across C}\}.$$

Integral version (divergence-flux form Green's Theorem): For $\mathbf{F} = \langle M(x,y), N(x,y)\rangle$

$$\iint_R \text{div}\,\mathbf{F}\ dA = \oint_{\mathbf{C}} \mathbf{F}\bullet\mathbf{n}\ ds = \text{flux across C}$$

Version 2 Green's Theorem: Curl and Circulation

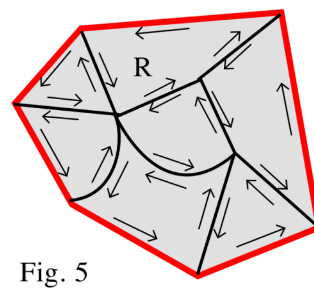


Fig. 5

Instead of looking at flow across edges of the cells, consider flow along each edge of a cell as in Fig. 5, and define the curl of each cell to be the sum of the flows around the edges of the cell counting flows in the counterclockwise direction to be positive and flows in the clockwise direction to be negative. On each inside shared edge (Fig. 6) the flow gets counted once as positive and once as negative so the sum of those two flows is 0. But this happens along every inside edge. If we add all of the curls together, the only flows that are not cancelled out in this way are the flows along the exterior edges of the collection, the flows along C (Fig. 7). This total flow around the boundary C of the collection is called the circulation.

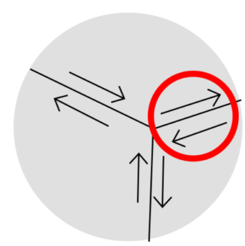


Fig. 6

In words: The circulation around the boundary C equals the sum of the circulations (curls) on the cells of R.

Finite version (Green's Theorem): $\sum_{C} \{\text{flow along outside edge}\} = \sum_{R} \text{curls dA}$

Integral version (curl-circulation form Green's Theorem): For $\mathbf{F} = \langle M(x,y), N(x,y)\rangle$

$$\oint_{\mathbf{C}} \mathbf{F}\bullet\mathbf{T}\ ds = \iint_R \text{curl}\,\mathbf{F}\ dA = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

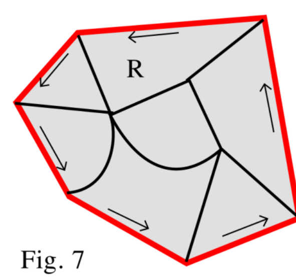


Fig. 7

If we could just take limits as all of the cells got smaller and smaller (it is not so easy) we would have both versions Green's Theorem which is discussed in Section 15.5:

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) = \oint_{\mathbf{C}} \mathbf{F}\bullet\mathbf{n}\ d = \int_C M\,dy - N\,dx = \text{flux}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \oint_{\mathbf{C}} \mathbf{F}\bullet\mathbf{T}\ ds = \int_C M\,dx + N\,dy = \text{circulation}$$

Problems

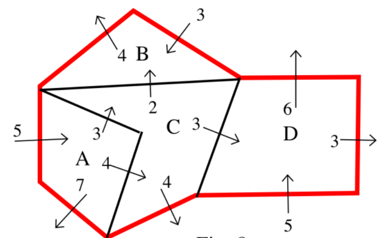


Fig. 8

1. For Fig. 8 verify that the sum of the divergences of all of the cells is equal to the flux across the boundary of the region.

2. For Fig. 9 verify that the sum of the curls of all of the cells is equal to the circulation around the boundary of the region.

Answer 1: A divergence = 9, B div = –1, C div = 2, D div = 1, so $\sum \text{div} = 11$.

Flux across the boundary = (7)+(4)+(–5)+(3)+(6)+(–3)+(4)+(–5)=11.

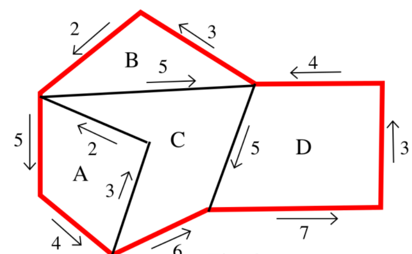


Fig. 9

Answer 2: A curl = 14, B curl = 10,

C curl =(6)+(–5)+(–5)+(–2)+(–3)= –9 , D curl = 19, so $\sum \text{curl} = 34$.

Circulation around the boundary = 34.

The following theorems of Stokes and Gauss extend Green's Theorem to higher dimensions.

Stokes' Theorem: Curl and Circulation

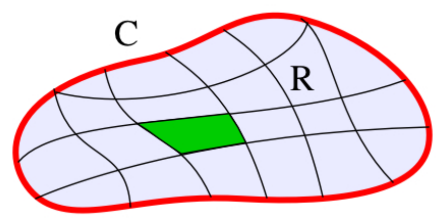
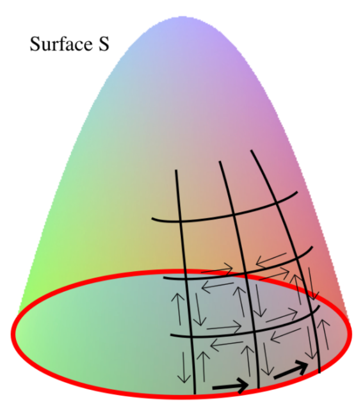


Fig. 10: Flat region R with boundary curve C

In Green's Theorem R was a planar region with boundary curve C (Fig. 10). Now suppose that the region R is a soap film and the boundary C is a rigid wire. If we gently blow on R to create an oriented, smooth surface with the same boundary C then each cell in the region R becomes a cell on the surface S. Just like in Green's Theorem, on each inside shared edge (Fig. 11) the flow gets counted once as positive and once as negative so the sum of those two flows is 0. But this happens along every inside edge. If we add all of the curls together, the only flows that are not cancelled out in this way are the flows along the exterior edges of the collection, the flows along the boundary C. This total flow around the boundary C is called the circulation.

Surface S



Boundary C

Fig. 11

In words: The circulation around the boundary C equals the sum of the circulations (curls) on the cells of surface S.

Finite version (Stokes' Theorem):

$$\sum_{C} \{\text{flow along outside edge}\} = \sum_{R} \text{curls } dA$$

Integral version (Stokes' Theorem): $$\oint_C \mathbf{F} \bullet \mathrm{T}\, ds = \oint_C \mathbf{F} \bullet \mathbf{dr} = \iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS$$

Gauss/Divergence Theorem: Flux and Divergence

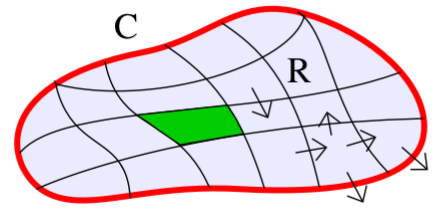


Fig. 12: Flat region R with boundary curve C

In Green's Theorem R was a planar region with boundary curve C, and the sum of the internal cell divergences was equal to the flux across the boundary C. Now suppose that instead of a region R in 2D there is a solid region E in 3D, a volume, and that the boundary (skin) of E is a surface S. Also imagine that E is partitioned into little 3D cells, and that each of these internal cells has a divergence, a net outward flow.

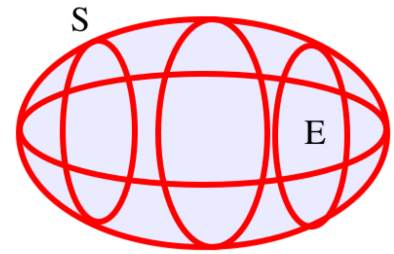


Fig. 13: E is the 3D region enclosed by the 2D boundary (skin) S

Just like in Green's Theorem, on each inside shared cell face the flow gets counted once as positive and once as negative so the sum of those two flows is 0. But this happens along every inside cell face. If we add all of the flows (divergences) for each cell together, the only flows that are not cancelled out in this way are the flows across the exterior faces of the collection, the flows across the boundary surface S. This total flow across the boundary S of the solid E is called the flux across S.

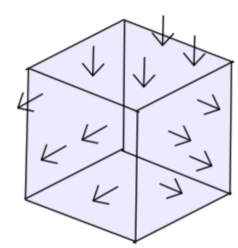


Fig. 14: cell in E

In words: The flux across the boundary S equals
the sum of the divergences on the cells of solid E.

Finite version (Gauss/Divergence Theorem):

$$\text{flux across S} = \sum_{S} \mathrm{F} \bullet \mathbf{n}\ dA = \sum_{E} \text{divs } dV$$

Integral Version: $$\text{flux across S} = \iint_{S} \mathrm{F} \bullet \mathbf{n}\ dA = \iiint_{E} \text{div } \mathbf{F}\ dV$$

Wrap up

In the following sections these theorems will be more carefully presented and partially proved, and we will actually do calculations using them. These are the final big three theorems of calculus, and they are both beautiful and very useful.