15.2 Del Operator and 2D Divergence and Curl

The divergence and curl of a vector field F describe two characteristics of the field at each point in the field. They will be extended into 3D in later sections, but they are more easily understood graphically and intuitively in 2D, so we start there.

Divergence: div **F**

Assume that the vector field F describes the flow of a liquid in 2D, perhaps water contained between two close-together parallel plates. The **divergence** is the rate per unit area that the water dissipates (**departs**, leaves) at the point P. A positive value for the divergence means that more water is leaving at P than is entering at P. But we can't really see a point so imagine a small circle C centered at P, and consider whether more water s leaving or entering this circle. Fig. 1(a) shows more water leaving than entering the circle around P so div **F**(P) > 0, Fig. 1(b) has more entering than leaving so div **F**(P) < 0, and Fig, 1(c) shows the same amount leaving as entering so div **F** (P) = 0.

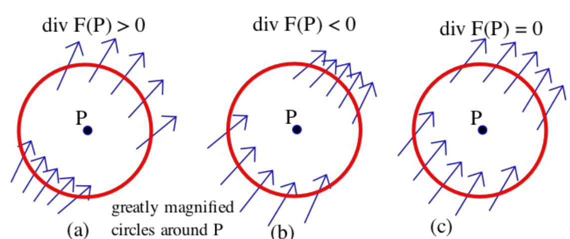


Fig. 1

Example 1: Estimate whether div **F**(P) is positive, negative or zero when P=A, B and C in Fig. 2.

Solution: (a) div F > 0 (b) div F = 0 (c) div F < 0

Practice 1: Estimate whether div **F**(P) is positive, negative or zero when P=A, B and C in Fig. 3.

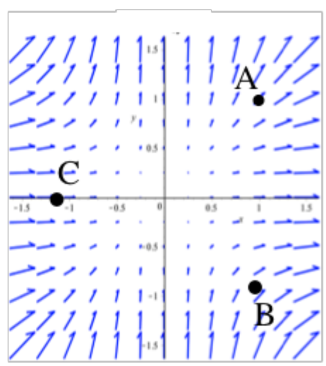


Fig. 2

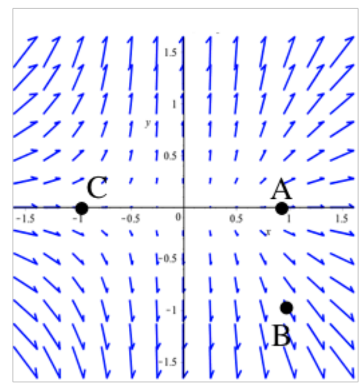


Fig. 3

It turns out that div **F**(P) is very easy to calculate.

> **Definition: Divergence** div F
>
> For a vector field $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ with continuous partial derivatives,
>
> the divergence of F at Point P is $\text{div } \mathbf{F}(P) = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$.
>
> (In 3D with $\mathbf{F}(x,y,z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, $\text{div } \mathbf{F}(P) = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z}$.)

We will justify this definition for divergence in section 15.10, but, for now, we will simply use it.

Example 2: The vector field in Fig. 2 is $\mathbf{F} = \langle x^2, y^2 \rangle$. Calculate div **F**(P) at P=A=(1,1), P=B=(1,–1) and P=C=(–1,0).

Solution: div $\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$ = 2x+2y. (a) at A div **F** =4, at B div **F** = 0, at C div **F** = –2.

Practice 2: The vector field in Fig. 3 is $\mathbf{F} = \langle \mathbf{x^2}, 2 \cdot y \rangle$. Calculate div **F**(P) at P=A=(1,0), P=B=(1,–1) and P=C=(–1,0).

For the radial field $\mathbf{F} = \langle x, y \rangle$, div $\mathbf{F} = 2$ at every point. For the radial field $\mathbf{F} = \langle -x, -y \rangle$, div $\mathbf{F} = -2$ at every point. And for the rotational field $\mathbf{F} = \langle -y, x \rangle$, div $\mathbf{F} = 0$ at every point. (Fig. 4)

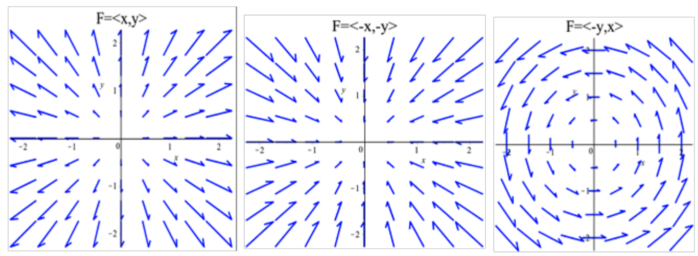


Fig. 4

Curl: curl F

The curl **F** (P) measures the rotation of the vector field **F** at the point P. Again picture a small circle centered at P, but this time imagine an axle at P and small paddles on the circle (Fig. 5). If the water vectors would rotate the little wheel counterclockwise (Fig. 6a) we say that curl **F** (P) > 0, if the rotation is clockwise (Fig. 6b) curl **F**(P) < 0, and if there is no rotation then curl **F** (P) = 0 (Fig. 6c)

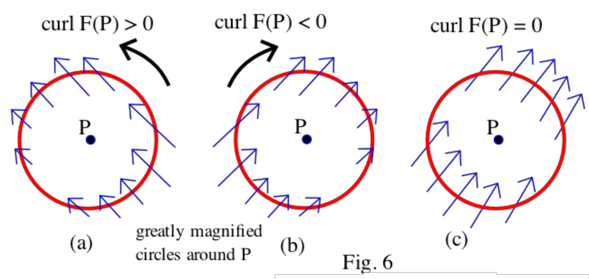


Fig. 6

Example 3: Estimate whether curl **F**(P) is positive, negative or zero when P=A, B and C in Fig. 7.

Solution: (a) at A curl **F** > 0, at B curl **F** < 0, at C curl **F** > 0

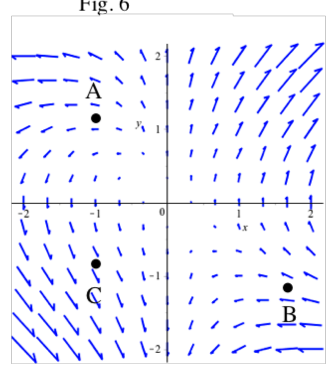


Fig. 7

> **Definition: Curl in 2D** curl F
>
> For a vector field $\mathbf{F}(x,y) = M\mathbf{i} + N\mathbf{j}$ with continuous partial derivatives,
>
> then the curl of F at Point P is curl $\mathbf{F}(P) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

Note: Later we will define curl F to be a vector quantity in 3D (still measuring a rotation) as

$$\text{curl F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$ For now we will just need and use the k component of this 3D vector. That will tell us about the rotation in the xy-plane,

Example 4: The vector field in Fig. 7 is $\mathbf{F} = \langle xy, x+y \rangle$ with A=(–1,1), B=(1.8,–1) and C=(–1,–1). Calculate curl **F**(P) at each point.

Solution: $\text{curl } \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - x$. At A curl **F** = 2, at B curl **F** = -0.8, at C curl **F** = 2.

Practice 3: Calculate curl **F** for the radial field $\mathbf{F} = \langle x, y \rangle$ and the rotational field $\mathbf{F} = \langle -y, x \rangle$.

If curl F=0 at all points in the field, then F is called **irrotational**.

The divergence and the curl measure completely different characteristics of the field at a point, and knowing the sign of one does not tell us anything about the sign of the other. For example, the curl of $\mathbf{F} = \langle x^2y + y, y^2 \rangle$ is $\text{curl } \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (0) - (x^2+1) < 0$ at all points, but $\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 2xy + 2y$ which can be positive, negative or zero depending on the location (x,y).

Practice 4: Find a vector field **F** so that div **F** > 0 at all points but curl **F** can be positive, negative or zero depending on the location (x,y).

del operator $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

A mathematical operator is like a function but it typically operates on functions or other advanced objects. For example, differentiation $\frac{d}{dx}[\ ]$ and integration $\int [\ ]dx$ are operators that do things to whatever function is put into the brackets. Similarly, the del operator does things to functions and even vector fields. And the del notation is a compact way to represent complicated operations (and an easy way to remember them).

in 2D: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ which we met earlier and called the 2D gradient of f

$\nabla \bullet F = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \bullet \langle M, N \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$ which is the 2D divergence, div **F**

$$\nabla \mathbf{x} F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$ which is the 2D curl, curl **F**

In 3D: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \right\rangle$ the 3D gradient of f

$\nabla \bullet F = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \bullet \langle M, N, P \rangle = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$ the 3D divergence, div **F**

$$\nabla \mathbf{x} \mathrm{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial y} \right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\mathbf{k} \quad \text{the 3D curl of F}$$

You should recognize the z-component of $\nabla \mathbf{x} \mathrm{F}$ as the 2D curl of F.

The del operator can be combined with itself to create new operations:

the Laplacian: $\nabla \bullet \nabla f = \left\langle \frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}, \frac{\partial}{\partial \mathbf{z}} \right\rangle \bullet \left\langle \frac{\partial f}{\partial \mathbf{x}}, \frac{\partial f}{\partial \mathbf{y}}, \frac{\partial f}{\partial \mathbf{z}} \right\rangle = \frac{\partial^2 f}{\partial \mathbf{x}^2} + \frac{\partial^2 f}{\partial \mathbf{y}^2} + \frac{\partial^2 f}{\partial \mathbf{z}^2}$

is important in mathematical physics.

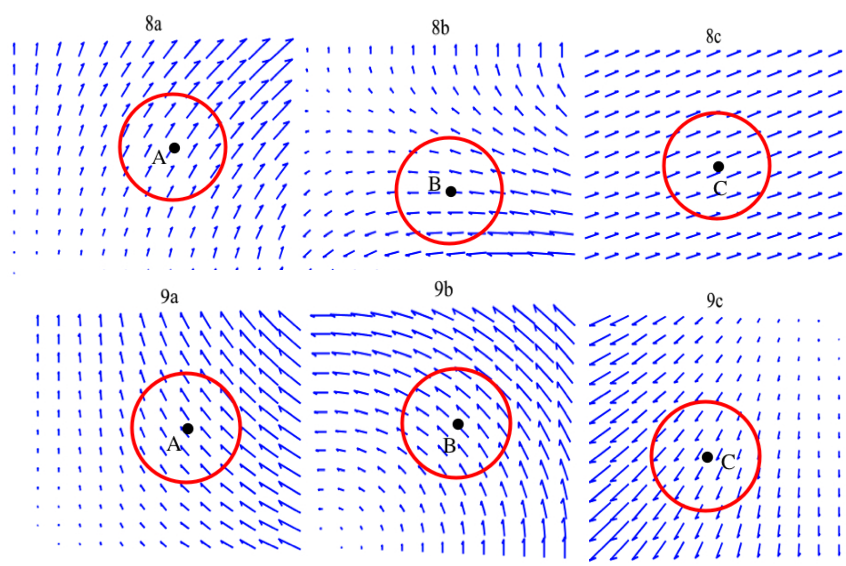
And the del operator can help illuminate connections between objects that seem unrelated:

$$\text{div curl F} = \nabla \bullet (\nabla \mathbf{x} \mathrm{F}) = \left\langle \frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \mathbf{y}}, \frac{\partial}{\partial \mathbf{z}} \right\rangle \bullet \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial y} \right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\mathbf{k} = \ \dots \ = 0$$

You can fill in the " …" by taking the dot product and then recognizing that the various mixed partial derivatives are equal (e.g, $\frac{\partial^2 f}{\partial \mathbf{x} \partial \mathbf{y}} = \frac{\partial^2 f}{\partial \mathbf{y} \partial \mathbf{x}}$).

Wrap up

The justifications for the definitions of the divergence and curl of a vector field at a point will come later in this chapter, but the ideas and calculations in 2D are relatively easy and they will help with the ideas and calculations when we get to Green's Theorem.



Problems

1. Estimate whether the div **F** is positive, negative or zero at A, B and C in Fig. 8.
2. Estimate whether the div **F** is positive, negative or zero at A, B and C in Fig. 9.

In problems 3 to 6, calculate div **F** at the given points.

3. $\mathbf{F} = \left\langle x^2 + 3y,\ 2y + x \right\rangle$ at A=(1, 3), B=(2, –1) and C=(–1, 3).
4. $\mathbf{F} = \left\langle x \cdot y^2,\ x^3 \cdot y + 3 \right\rangle$ at A=(1, 1), B=(2, –1) and C=(0, –2).
5. $\mathbf{F} = \left\langle 5x - 3y,\ x + 2y \right\rangle$ at A=(3, 2), B=(0, 3) and C=(1, 4).

6. $\mathbf{F} = \langle x^2 - y^2, x^2 + y^2 \rangle$ at A=(2, 3), B=(-2, 2) and C=(3, –4).

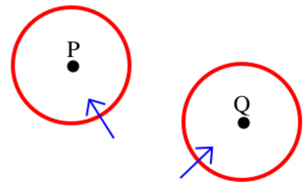


Fig. 10

7. In Fig. 10 add additional vectors so div $\mathbf{F}(P) > 0$.
8. In Fig. 10 add additional vectors so div $\mathbf{F}(Q) < 0$.

9. Estimate whether the curl **F** is positive, negative or zero at A, B and C in Fig. 8.
10. Estimate whether the curl **F** is positive, negative or zero at A, B and C in Fig. 9.

In problems 11 to 14, calculate curl **F** at the given points.

11. $\mathbf{F} = \langle x^2 + 3y, 2y + x \rangle$ at A=(1, 3), B=(2, –1) and C=(–1, 3).
12. $\mathbf{F} = \langle x \cdot y^2, x^3 \cdot y + 3 \rangle$ at A=(1, 1), B=(2, –1) and C=(0, –2).
13. $\mathbf{F} = \langle 5x - 3y, x + 2y \rangle$ at A=(3, 2), B=(0, 3) and C=(1, 4).
14. $\mathbf{F} = \langle x^2 - y^2, x^2 + y^2 \rangle$ at A=(2, 3), B=(–2, 2) and C=(3, –4).

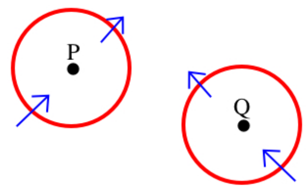


Fig. 11

15. In Fig. 11 add additional vectors so curl $\mathbf{F}(P) > 0$.
16. In Fig. 11 add additional vectors so curl $\mathbf{F}(Q) < 0$.

17. Show that the div and curl are always 0 for a constant field $\mathbf{F} = \langle a, b \rangle$.
18. Show that the div and curl are constant for a linear field $\mathbf{F} = \langle ax + by, cx + dy \rangle$.
19. If a rotational field is flowing counterclockwise, does that mean the curl at each point is positive. Justify your conclusion.
20. If the y-component of **F** is always 0, what can you conclude about the divergence?
21. If the y-component of **F** is always 0, what can you conclude about the curl?

Practice Answers

Practice 1: (a) div $F > 0$ (b) div $F = 0$ (c) div $F < 0$

Practice 2: div $\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 2x + 2$. (a) at A div $\mathbf{F} = 4$, at B div $\mathbf{F} = 4$, at C div $\mathbf{F} = 0$.

Practice 3: For $\mathbf{F} = \langle x, y \rangle$ curl $\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ at every point. No rotation anywhere.

For $\mathbf{F} = \langle -y, x \rangle$ curl $\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (1) - (-1) = 2$ at every point. The rotation is counterclockwise at every point.

Practice 4: You need a field $\mathbf{F} = \langle M, N \rangle$ so that $\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} > 0$ everywhere but so $\text{curl } \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ contains variables. $\mathbf{F} = \langle x^3 + x + y^2, y^3 \rangle$ works.

$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 3x^2 + 1 + 3y^2 > 0$ and $\text{curl } \mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (0) - (2y)$.