14.4 APPLICATIONS OF DOUBLE INTEGRALS

In Section 5.4 we used single integrals to determine the mass, moments about each axis, and the center of mass of a thin plate (lamina) with uniform density δ. In this section we will use double integrals to extend those ideas and calculations to thin plates with varying densities.

Uniform density for a 2D region between f and the x-axis for a ≤ x ≤ b

Total Mass: $M = \delta \cdot \int_a^b f(x)\ dx$

Moments: about y-axis $M_y = \delta \cdot \int_a^b x \cdot f(x)\ dx$ about x-axis $M_x = \delta \cdot \int_a^b \frac{1}{2} f^2(x)\ dx$

Center of mass: $\bar{x} = \frac{M_y}{M}$, $\bar{y} = \frac{M_x}{M}$

If the density of the plate depends on the location (x,y) on the plate, then δ is a function of x and y: $\delta(x,y)$ with units of the form (mass)/area such as kg/m^2 .

Nonuniform density for a 2D region R in the xy-plane

Area: $Area = \iint_R 1\ dA$

Total Mass: $M = \iint_R \delta(x,y)\ dA$

Moments: about y-axis $M_y = \iint_R x \cdot \delta(x,y)\ dA$ about x-axis $M_x = \iint_R y \cdot \delta(x,y)\ dA$

Center of mass: $\bar{x} = \frac{M_y}{M}$, $\bar{y} = \frac{M_x}{M}$

Note: Be careful to use the x with M_y since the little dA piece is x units from the y-axis. Similarly, use the y with the M_x formula since the little dA piece is y units from the x-axis.

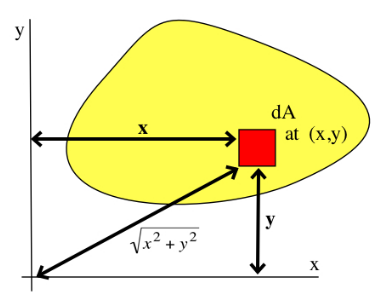

Fig. 1

Example 1: $R = \{(x,y) : 0 \le x \le 2 \text{ and } 0 \le y \le 4 - x^2\}$ (Fig. 2).

(a) Determine the center of mass of R if the region has uniform density δ.

(b) Determine the center of mass of R if $\delta(x,y) = x \cdot y^2$.

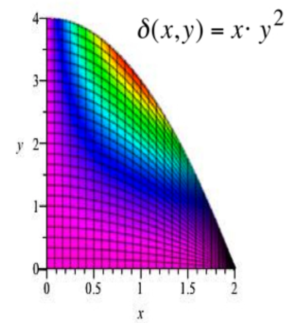

Fig. 2

Solution: (a) Using double integrals,

$$M = \iint_R \delta(x,y)\ dAM = \int_{x=0}^{2} \int_{y=0}^{4-x^2} 1\ dy\ dx = \frac{16}{3}.$$

$$M_y = \int_{x=0}^{2} \int_{y=0}^{4-x^2} x\ dy\ dx = 4 \text{ and } M_x = \int_{x=0}^{2} \int_{y=0}^{4-x^2} y\ dy\ dx = \frac{128}{15}$$

so $\bar{x} = \frac{3}{4}$ and $\bar{y} = \frac{8}{5}$.

(b) $\delta(x,y) = x \cdot y^2$, $M = \int_{x=0}^{2} \int_{y=0}^{4-x^2} x \cdot y^2\ dy\ dx = \frac{32}{3}$, $M_y = \int_{x=0}^{2} \int_{y=0}^{4-x^2} x \cdot x \cdot y^2\ dy\ dx = \frac{8192}{945}$

and $M_x = \int_{x=0}^{2} \int_{y=0}^{4-x^2} y \cdot x \cdot y^2\ dy\ dx = \frac{128}{5}$ so $\bar{x} = \frac{256}{315}$ and $\bar{y} = \frac{12}{5}$.

Practice 1: $R = \{(x,y) : 0 \le x \le 2 \text{ and } 0 \le y \le 4 - x^2\}$. Determine the center of mass of R if $\delta(x,y) = x^2 \cdot y$.

Moments of Inertia (second moments) and Radii of Gyration

Moments of inertia are needed to calculate the kinetic energy of rotating objects and also for formulas for stiffness of beams. Note that the second moments formulas are very similar to the first moment formulas but that they use the squares x^2 and y^2 of the lever arms distances of the dA piece from the axes instead of just x and y.

Moments of Inertia (second moments) of region R in the xy-plane

About the x-axis: $I_x = \iint_R y^2 \cdot \delta(x,y)\ dA$ About the y-axis $I_y = \iint_R x^2 \cdot \delta(x,y)\ dA$

About the origin: $I_0 = \iint_R (x^2 + y^2) \cdot \delta(x,y)\ dA = I_x + I_y$

Radii of Gyration of region R in the xy-plane

About the x-axis: $R_x = \sqrt{\frac{I_x}{M}}$ About the y-axis: $R_y = \sqrt{\frac{I_y}{M}}$ About the origin: $R_0 = \sqrt{\frac{I_0}{M}}$

The moment of inertia about the origin, I_0, is also called the **polar moment**, and that integral uses the square of the distance of the dA piece from the origin.

The radius of gyration R_x is the distance from the x-axis that a point with mass M must be in order to give the same moment of inertia I_x: $I_x = M \cdot R_x^2$.

Example 2: If the units of x and y are meters and the units of $\delta(x,y)$ are kg/m^2, determine the units for I_x and R_x.

Solution: dA = $dx \cdot dy$ has units m^2, so $I_x = \iint_R y^2 \cdot \delta(x,y)\ dA$ has units $m^2 \cdot \frac{kg}{m^2} \cdot m^2 = kg \cdot m^2$.

$$R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{kg \cdot m^2}{kg}} = m.$$

Practice 2: If the units of x and y are feet and the units of $\delta(x,y)$ are $\frac{slug}{ft^2}$, determine the units for I_x and R_x. (One slug of mass in the British system is approximately 14.594 kg.)

Example 3: Suppose x and y are given in meters and the triangular region R={(x,y): 0≤x≤2 and 0≤y≤4-2x} (Fig. 3) has uniform density $\delta(x,y) = \delta\ \frac{kg}{m^2}$. Calculate M, $\bar{x}$, I_x, and R_x.

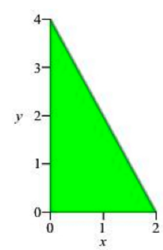

Fig. 3

Solution: $M = \iint_R \delta\ dA = \int_0^2 \int_0^{4-2x} \delta\ dy\ dx = \int_0^2 \delta \cdot (4-2x)\ dx = 4\delta$ kg.

$$M_y = \iint_R x \cdot \delta\ dA = \int_0^2 \int_0^{4-2x} x \cdot \delta\ dy\ dx = \int_0^2 \delta(4x - 2x^2)\ dx = \frac{8}{3}\delta\ kg \cdot m$$

$$I_x = \iint_R y^2 \cdot \delta\ dA = \int_0^2 \int_0^{4-2x} y^2 \cdot \delta\ dy\ dx = \int_0^2 \frac{\delta}{3}(4-2x)^3\ dx = \frac{32}{3}\ kg \cdot m^2$$

so $\bar{x} = \frac{M_y}{M} = \frac{2}{3}\ m$ and $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{8}{3}\ m^2} \approx 1.63\ m$.

Practice 3: Calculate $\bar{y}$, I_y, and R_y for the region R in Example 3.

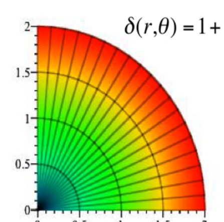

Fig. 4

Example 4: R is the quarter circle of radius 2 in the first quadrant with density $\delta(r,\theta) = 1 + r$. (Fig. 4) Calculate M, $\bar{x}$, I_x, and R_x.

Solution: $$M = \iint_R \delta\ dA = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} (1+r) \cdot r\ dr\ d\theta = \int_0^{\pi/2} \frac{14}{3}\ d\theta = \frac{7}{3}\pi$$

$$M_y = \iint_R x \cdot \delta \; dA = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} r \cdot \cos(\theta) \cdot (1+r) \cdot r \; dr \; d\theta = \int_0^{\pi/2} \frac{20}{3}\cos(\theta) \; d\theta = \frac{20}{3}$$

$$I_x = \iint_R y^2 \cdot \delta \; dA = \int_{\theta=0}^{\pi/2} \int_{r=0}^{2} r^2 \cdot \sin^2(\theta) \cdot (1+r) \cdot r \; dr \; d\theta = \int_0^{\pi/2} \frac{52}{5}\sin^2(\theta) \; d\theta = \frac{13}{5}\pi$$

so $\bar{x} = \frac{M_y}{M} = \frac{20}{7\pi} \approx 0.909$ and $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{39}{35}} \approx 1.056$.

Note: Because of the symmetry of the R and δ, then $\bar{y} = \bar{x}$, $I_y = I_x$, and $R_y = R_x$.

Problems

In problems 1 to 8 use double integrals to calculate the area, total mass, the moments about each axis, and the center of mass of the region. Plot the location of the center of mass on the region.

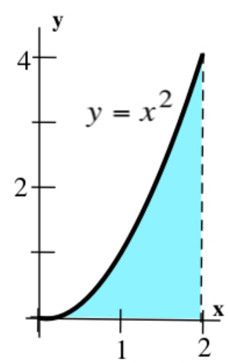

Fig. 5

1. R is the rectangular region bounded by the x and y axes and the lines x=2 and y=4 with $\delta = 1$.

2. R is the rectangular region bounded by the x and y axes and the lines x=2 and y=4 with $\delta = xy$.

3. R is the shaded region in Fig. 5 and $\delta(x,y) = 1 + x$.

4. R is the shaded region in Fig. 5 and $\delta(x,y) = 1 + y$.

5. R={(x,y): 0≤x≤3, 0≤y≤1+x} and $\delta(x,y) = x + y$.

6. R={(x,y): 0≤x≤3, 0≤y≤1+x} and $\delta(x,y) = 1 + y$.

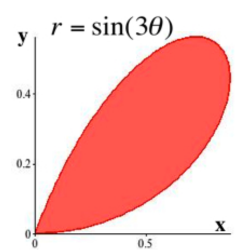

Fig. 6

7. R is the shaded region in Fig. 6 and $\delta(r,\theta) = 1$.

8. R is the shaded region in Fig. 6 and $\delta(r,\theta) = r$.

9. R={(r, θ): $0 \le \theta \le \pi$, $0 \le r \le 1 + \cos(\theta)$} and $\delta(r,\theta) = r$. (Fig. 7)

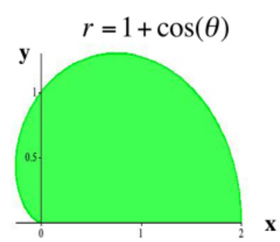

Fig. 7

10. R={(r, θ): $0 \le \theta \le \pi$, $0 \le r \le 1 + \cos(\theta)$} and $\delta(r,\theta) = 1 + r$. (Fig. 7)

11. Calculate the values of I_x and R_x for the region in Problem 1.

12. Calculate the values of I_y and R_y for the region in Problem 1.

13. Calculate the values of I_x and R_x for the region in Problem 2.

14. Calculate the values of I_y and R_y for the region in Problem 2.

In Problems 15 to 20, use the fact that the attractive gravitational force between two points with masses M and m at a distance of r is $\text{force} = \frac{\text{GMm}}{\text{r}^2}$. (It helps to sketch the regions.)

15. Represent the total force between a point mass of 10 kg at the origin and a bar from x=2 to x=4 m on the x-axis with a mass of 8 kg. (This is a single integral.)

16. 15. Represent the total force between a point mass of M kg at the origin and a bar from a to b m (0<a<b) m on the x-axis with a mass of m kg. (This is a single integral.)

17. Represent the total force between a bar along the x-axis from 0 to 2 with mass of 10 kg and another bar on the x-axis from 4 to 7 with a mass of 9 kg. (This is a double integral.)

18. Represent the total force between a bar along the x-axis from a to b with total mass M and another bar on the x-axis from c to d with a mass of m (a<b<c<d). (This is a double integral.)

Practice Solutions

Practice 1: $R = \{(x,y) : 0 \le x \le 2 \text{ and } 0 \le y \le 4 - x^2\}$. Determine the center of mass of R if $\delta(x,y) = x^2 \cdot y$.

$$(\delta(x,y) = x^2 \cdot y\ ,\ M = \int_{x=0}^{2} \int_{y=0}^{4-x^2} x^2 \cdot y\ dy\ dx = \frac{512}{105},\ M_y = \int_{x=0}^{2} \int_{y=0}^{4-x^2} x \cdot x^2 \cdot y\ dy\ dx = \frac{16}{3}$$

$$\text{and } M_x = \int_{x=0}^{2} \int_{y=0}^{4-x^2} y \cdot x^2 \cdot y\ dy\ dx = \frac{8192}{945} \quad \text{so } \bar{x} = \frac{35}{32} \text{ and } \bar{y} = \frac{16}{9} .$$

Practice 2: dA = $dx \cdot dy$ has units ft^2, so $I_x = \iint_R y^2 \cdot \delta(x,y)\ dA$ has units $ft^2 \cdot \frac{slug}{ft^2} \cdot ft^2 = slug \cdot ft^2$.

$$M = \iint_R \delta(x,y)\ dA = \frac{slug}{ft^2} \cdot ft^2 = slug \text{ so } R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{slug \cdot ft^2}{slug}} = feet.$$

Practice 3: $M_x = \iint_R y \cdot \delta\ dA = \int_0^2 \int_0^{4-2x} y \cdot \delta\ dy\ dx = \int_0^2 \delta \cdot \frac{1}{2}(4-2x)^2\ dx = \frac{16}{3}\delta$

so $\bar{y} = \frac{16/3\ \delta}{4\ \delta} = \frac{4}{3}\ m$.

$$I_y = \iint_R x^2 \cdot \delta\ dA = \int_0^2 \int_0^{4-2x} x^2 \cdot \delta\ dy\ dx = \int_0^2 \delta \cdot x \cdot (4-2x)\ dx = \frac{8}{3}\delta\ kg \cdot m^2$$

so $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{8/3\ \delta}{4\ \delta}\ m^2} = \sqrt{\frac{2}{3}} \approx 0.816\ m$