14.2   DOUBLE INTEGRALS OVER GENERAL REGIONS

The double integrals over rectangular regions in the section 14.1 are relatively straightforward, but many applied situations have domains that are not rectangular, and this section considers those. In section 14.1 our focus was on finding antiderivatives (twice) and evaluating the double integrals. In this section the main difficulties involve setting up the endpoints of the integrals. This is very important and is vital for later sections.

The starting ideas here are the same as in 14.1:

* partition the general domain D into small Δx_i by Δy_j rectangles, (Fig. 1)

* create a double Riemann sum of the areas of the boxes above each rectangle,

$$\sum_j \sum_i f(x_i, y_j) \cdot \Delta x_i \cdot \Delta y_j = \sum_j \sum_i f(x_i, y_j) \cdot \Delta A_{ij}$$

* take the limits as all of the Δx_i and Δy_j values approach 0

   so they fill the domain, and

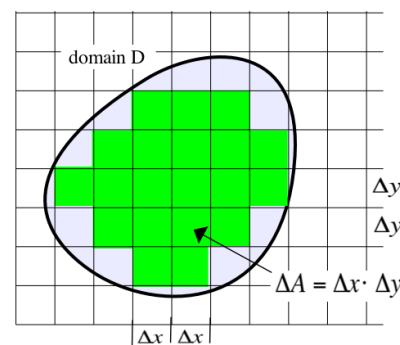* create a double integral $\iint\limits_D f(x,y) \, dA$.

Fig. 1

The notation $\iint\limits_D f(x,y) \, dA$ is neat and compact and is commonly used, but in order to compute the value of the double integral we need to explicitly

rewrite it as $\iint\limits_D f(x,y) \, dx \, dy$ or $\iint\limits_D f(x,y) \, dy \, dx$ with the appropriate endpoints of integration.

Note: All of our work with double integrals assumes that the domain D is "bounded" (fits inside a finite rectangle) and that the boundary curve of D is "nice" (consists of a finite number of smooth curves with finite length). In most applications, these are not major restrictions.

Two Easiest Cases

Case V: For each x* value, D contains a single vertical segment of y values (Fig. 2).

Case H: For each y* value, D contains a single horizontal segment of x values .

Other: It is possible for the domain D to be both Case V and Case H or neither.

Fig. 2 — Case V, Case H, Neither
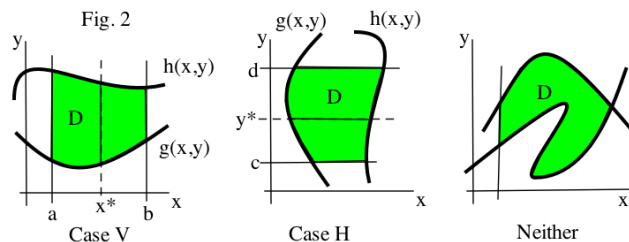
Case V: For each value of x between a and b, D consists of those y with $g(x) \leq y \leq h(x)$. For each value of x,
we only care about f(x,y) for y values between g(x) and h(x).

Then $\iint\limits_{D} f(x,y) \, dA = \int\limits_{x=a}^{x=b} \int\limits_{y=g(x)}^{y=h(x)} f(x,y) \, dy \, dx$ (or simply $\int\limits_{a}^{b} \int\limits_{g(x)}^{h(x)} f(x,y) \, dy \, dx$).

Example 1: $f(x,y) = 1 + 4x + 2y$ and D is the region in Fig. 3. Calculate the
volume of the solid between D and the surface z=f(x,y).

Solution: For each x between 0 and 2, the y values are between $2 + x$

and x^2 so Volume $= \iint\limits_{D} f(x,y) \, dA = \int\limits_{x=0}^{x=2} \int\limits_{y=x^2}^{y=2+x} 1 + 4x + 2y \, dy \, dx$.   Fig. 3

We deal with this double integral in the same way we did in section 14.1 by starting
with the inside integral.

$$\int\limits_{y=x^2}^{y=2+x} 1 + 4x + 2y \, dy = y + 4xy + y^2 \; |_{y=x^2}^{y=2+x}$$

$$= \{(2+x) + 4x(2+x) + (2+x)^2\} - \{(x^2) + 4x(x^2) + (x^2)^2\} = -x^4 - 4x^3 + 4x^2 + 13x + 6 .$$

Then $\int\limits_{x=0}^{x=2} \{-x^4 - 4x^3 + 4x^2 + 13x + 6\} \, dx = \dfrac{394}{15} \approx 26.27$

If the units of x, y and z=f(x,y) are centimeters, then the double integral units are cm^3 .

Practice 1: $f(x,y) = 2 + x + 2y$ and D is the smiley region in Fig. 4. Calculate

$\iint\limits_{D} f(x,y) \, dA = $ volume of the solid between D and the surface z=f(x,y).

Fig. 4

Example 2: Determine the integration endpoints for

$\iint\limits_{D} f(x,y) \, dy \, dx$ on the domain D shown in Fig. 5.

Solution: Since the dx is on the outside, the endpoints of the outside
integral will be from x=0 to x=2, But for each x value between 0 and 2 (Fig.
6), a vertical slice at x enters the domain D when y=2x (at the bottom) and
exits when y=4 (at the top) so the endpoints of the inside integral are y=2x and

y=4: $\int\limits_{x=0}^{x=2} \int\limits_{y=2x}^{y=4} f(x,y) \, dy \, dx$ or simply $\int\limits_{0}^{2} \int\limits_{2x}^{4} f(x,y) \, dy \, dx$.

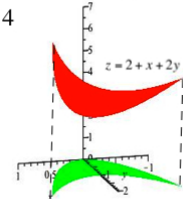Fig. 5

finish of inside
integral is y=4

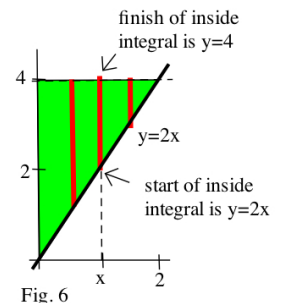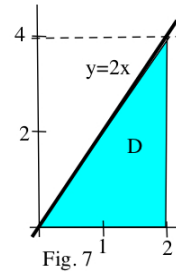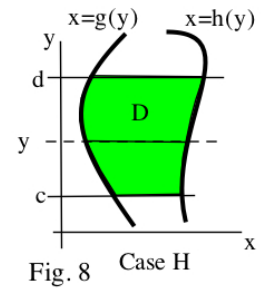start of inside
integral is y=2x
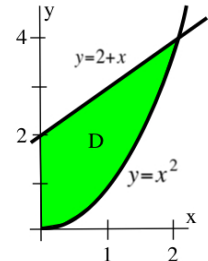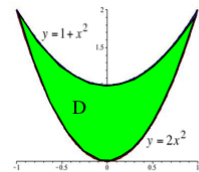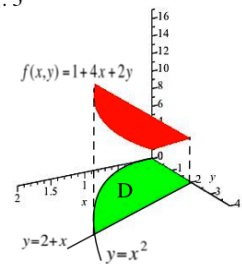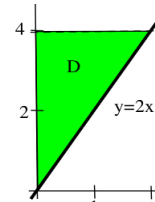
Fig. 6

Practice 2: Determine the integration endpoints for $\iint\limits_{D} f(x,y)\, dy\, dx$ on

the domain D shown in Fig. 7.

Case H: For each value of y between c and d, the domain D consists of those x values

with $g(y) \le x \le h(y)$. For each value of y, we only care about f(x,y) for x
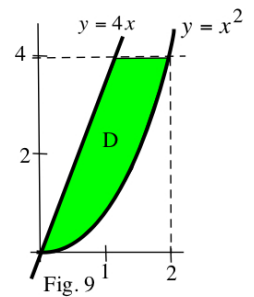
values between g(y) and h(y).

Then $\iint\limits_{D} f(x,y)\, dA = \int\limits_{y=c}^{y=d} \int\limits_{x=g(y)}^{x=h(y)} f(x,y)\, dx\, dy$ (or simply

$\int\limits_{c}^{d} \int\limits_{g(y)}^{h(y)} f(x,y)\, dx\, dy$).

Note: If the boundary function has the form y=f(x) then we need to solve for x as a

function of y. For example, if the boundary is y = f(x) = 1+2x, then we need to
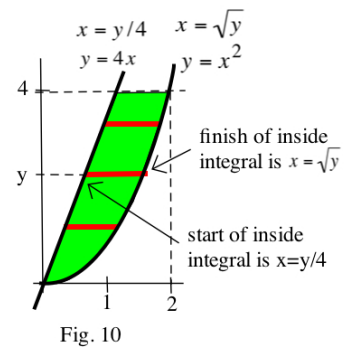
solve for x to get x = g(y) = (y-1)/2.

Example 3: Determine the integration endpoints for $\iint\limits_{D} f(x,y)\, dx\, dy$ on

the domain D shown in Fig. 9.

Solution: Clearly the horizontal slices go from y = 0 to y = 4 so those are the

endpoints of the outside integral. Then we need to convert the y=4x and

$y = x^2$ from functions of x to functions of y: x = y/4 and $x = \sqrt{y}$. For

each y value between 0 and 4 we need to see when a horizontal slice

at y enters and exits the domain (Fig. 10).

The result is $\iint\limits_{D} f(x,y)\, dx\, dy = \int\limits_{y=0}^{y=4} \int\limits_{x=y/4}^{x=\sqrt{y}} f(x,y)\, dx\, dy$.

Practice 3: Determine the integration endpoints for $\iint\limits_{D} f(x,y)\, dx\, dy$ on

the domain D shown in Fig. 11.
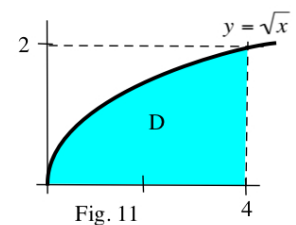
Splitting the Domain & Switching the Order of Integration

Sometimes the domain requires that we do the double integral in two (or more) pieces.

Example 4: Determine the integration endpoints for $\iint\limits_{D} f(x,y)\ dy\ dx$ on
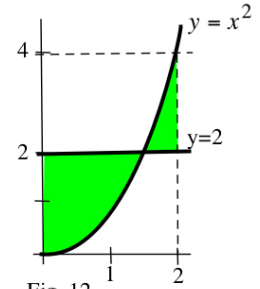
the shaded domain D shown in Fig. 12.

Fig. 12

Solution: For 0≤x≤1, the bottom boundary is $y = x^2$ and the top boundary is y=2.

For 0<x≤2 the top and bottom boundary functions are switched sp we need to use two double

integrals:

$$\iint\limits_{D} f(x,y)\ dy\ dx\ =\ \int\limits_{x=0}^{x=1}\int\limits_{y=x^2}^{y=2} f(x,y)\ dy\ dx + \int\limits_{x=1}^{x=2}\int\limits_{y=2}^{y=x^2} f(x,y)\ dy\ dx$$

Practice 4: Determine the integration endpoints for $\iint\limits_{D} f(x,y)\ dx\ dy$ on

the shaded domain D shown in Fig. 12. Note the switch to dx dy in the double integral --

this is also an example of switching the order of integration.

Example 5:   Switch the order of integration of $\int\limits_{x=-2}^{x=2}\int\limits_{y=x^2}^{y=4} f(x,y)\ dy\ dx\,.$

Fig. 13

Solution: From the given double integral we can create a picture of the domain

(Fig. 13) and then write the double integral with the order dx dy:

$$\int\limits_{x=-2}^{x=2}\int\limits_{y=x^2}^{y=4} f(x,y)\ dy\ dx\ =\ \int\limits_{y=0}^{y=4}\int\limits_{x=-\sqrt{y}}^{x=+\sqrt{y}} f(x,y)\ dx\ dy\ .$$
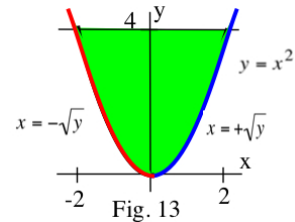
Why are we doing this?

Sometimes one order for a double integral is much easier to evaluate than the other order, and then it is

worthwhile to be able to convert to the easier way.

Example 6: Evaluate $\iint\limits_{R} e^{\left(x^2\right)}\ dA$ for R = {region for y between 0 and 1 and x between 3y and 3}.

Solution: It is natural to set up the integrals as $\displaystyle\int_{y=0}^{1}\int_{x=3y}^{3} e^{\left(x^2\right)}\ dx\ dy$ since that is the description of the

region R, but as soon as we try the inside integral $\displaystyle\int_{x=3y}^{3} e^{\left(x^2\right)}\ dx$ we are stuck because $e^{\left(x^2\right)}$ does

not have an elementary antiderivative. Fig. 14 shows the region R

When we switch the order the integrals become $\displaystyle\int_{x=0}^{3}\int_{y=0}^{x/3} e^{\left(x^2\right)}\ dy\ dx$.

Fig. 14
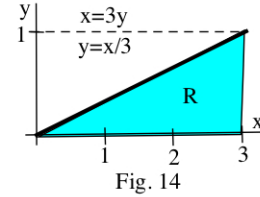
Now things are easier: $e^{\left(x^2\right)}$ does not depend on y so

$\displaystyle\int_{y=0}^{x/3} e^{\left(x^2\right)}\ dy = y\cdot e^{\left(x^2\right)}\ \Big|_{y=0}^{y=x/3} = \frac{1}{3} y\cdot e^{\left(x^2\right)}.$ Then the outside integral is $\displaystyle\int_{x=0}^{3} \frac{1}{3} x\cdot e^{\left(x^2\right)}\ dx$.

With the simple change of variable $u = x^2$ and $du = 2x\ dx$ we get

$\displaystyle\int \frac{1}{6} e^{(u)}\ du = \frac{1}{6} e^{(u)} = \frac{1}{6} e^{\left(x^2\right)}\ \Big|_{x=0}^{x=3} = \frac{1}{6} e^9 - \frac{1}{6}.$

Practice 5: Which is easier? (a) $\displaystyle\int_{x=0}^{1}\int_{y=x}^{1} \cos(y^2)\ dy\ dx$ or (b) $\displaystyle\int_{y=0}^{1}\int_{x=0}^{y} \cos(y^2)\ dx\ dy$

Properties of Double Integrals

These properties all are similar to the familiar properties of single integrals and follow from the properties of summations. We will not prove them here and have already used them in the examples. You should understand what each of them means geometrically.

(1) $\displaystyle\iint_{D} c\cdot f(x,y)\ dA = c\cdot\iint_{D} f(x,y)\ dA$

(2) $\displaystyle\iint_{D} \{f(x,y) + g(x,y)\}\ dA = \iint_{D} f(x,y)\ dA + \iint_{D} g(x,y)\ dA$

(3)   If $g(x,y) \le f(x,y)$ for all (x,y) in D, then $\displaystyle\iint_{D} g(x,y)\ dA \le \iint_{D} f(x,y)\ dA$

(4) $\displaystyle\iint_{D} 1\ dA = \text{area}(D)$

(5)   If $m \le f(x,y) \le M$ for all (x,y) in D, then $\displaystyle m\cdot\text{area}(D) \le \iint_{D} f(x,y)\ dA \le M\cdot\text{area}(D)$.

(6)   If D1 and D2 are disjoint domains and D = D1 ∪ D2 (= all (x,y) in D1 or D2)

then $\iint\limits_{D} f(x,y) \, dA = \iint\limits_{D1} f(x,y) \, dA + \iint\limits_{D2} f(x,y) \, dA$.

In this section the focus has been on setting up the endpoints of integration for non-rectangular domains. Some computer programs can evaluate double integrals, but only after the user has determined the endpoints. Even with technology you need to be able to determine those endpoints. And since it is sometimes much easier to integrate in one order than the other, it is important that you be able to write the double integral in either order.

Using Technology -- MAPLE

Some of the double integrals that appear in later sections and many of them in applications are very difficult to evaluate "by hand." Fortunately technology can be very helpful. The computer program MAPLE is very powerful and relatively simple to use.

In Example 1 we were able to evaluate the double integral of $f(x,y) = 1 + 4x + 2y$ on the region

$R = \{(x,y): 0 \le x \le 2, x^2 \le y \le 2 + x\}$ by hand, but the MAPLE command

   int(1+4*x+2*y, y=x^2 .. 2+x, x=0 ..1);

gives the same result, $\dfrac{379}{30}$, much quicker. And MAPLE can evaluate much more difficult integrals such as

$\int\limits_{x=0}^{1} \int\limits_{y=0}^{x+2} xy + \cos(xy) \; dy \; dx$. The MAPLE command   int(x*y+cos*x*y), y=0 .. x+2, x=0 .. 1.0) : quickly

gives 3.387845686 .

MAPLE syntax: int(formula for f(x,y) , y= lower endpoint .. upper endpoint, x= lower x .. upper x) ;

The program Mathematica and the online WolframAlpha can also evaluate double integrals.

PROBLEMS

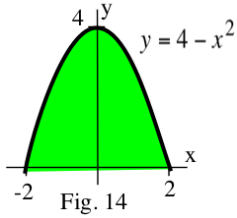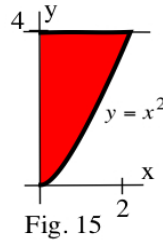For problems 1 – 6 sketch the domain of integration and evaluate the integrals.

1. $\int\limits_{x=0}^{x=3} \int\limits_{y=x}^{y=2x} x \; dy \; dx$

2. $\int\limits_{x=0}^{x=2} \int\limits_{y=0}^{y=2x} 3 + 4x + 2y \; dy \; dx$

3. $\int\limits_{y=1}^{y=4} \int\limits_{x=y}^{x=2y} 7 + 3x - 4y \; dx \; dy$

4. $\int\limits_{x=0}^{x=1} \int\limits_{y=x^2}^{y=x} x^2 + \sqrt{y} \; dy \; dx$

5. $\int\limits_{x=0}^{x=1} \int\limits_{y=0}^{y=x} \sin(x^2) \; dy \; dx$

6. $\int\limits_{y=0}^{y=2} \int\limits_{x=\sqrt{y}}^{x=3} x + y^2 \; dx \; dy$

In problems 7-12 a shaded domain D is shown. Determine the endpoints for both $\iint\limits_{D} f\, dy\, dx$ and $\iint\limits_{D} f\, dx\, dy$.

7.   Fig. 14   8.   Fig. 15   9.   Fig. 16
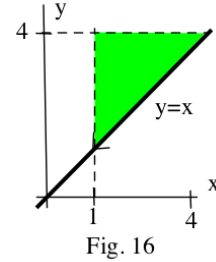

Fig. 14

Fig. 15

Fig. 16

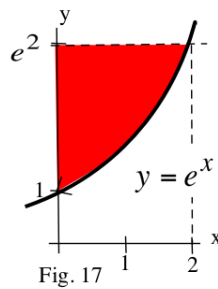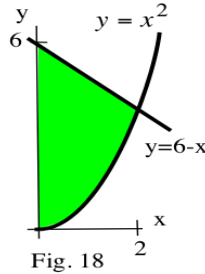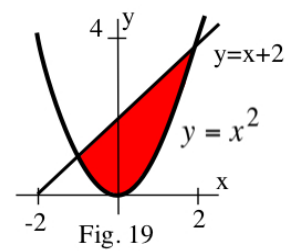10. Fig. 17   11. Fig. 18   12.

Fig. 17

Fig. 18

Fig. 19

In problems 13 - 20 sketch the domain of integration, set up the appropriate double integral and evaluate it.

13.. $\iint\limits_{D} xy\, dA$ where $D = \{ (x,y) : 0 \le x \le 1, x^2 \le y \le \sqrt{x} \ \}$.

14. $\iint\limits_{D} (x+y)\, dA$ where $D = \{ (x,y) : \pi/6 \le x \le \pi/4, \sin(x) \le y \le \cos(x) \}$.

15. $\iint\limits_{D} \left(y - xy^2 \right)\, dA$ where $D = \{ (x,y) : -y \le x \le 1+y, 0 \le y \le 1 \}$.

16. $\iint\limits_{D} x^2 y\, dA$ where $D = \{ (x,y) : x^2 \le y \le 2+x, -1 \le x \le 2 \}$.

17. $\iint\limits_{D} 1 + x \cdot \cos(y)\, dA$ where D is the bounded region between $y = 0$ and $y = 4 - x^2$.

18. $\iint\limits_{D} \left(x^2 + y \right)\, dA$ where D is the bounded region between $y = x^2$ and $y = 8 - x^2$.

19. $\iint\limits_{D} 4y^3\, dA$ where D is the bounded region between $y = 2 - x$ and $y^2 = x$.

20. $\iint\limits_{D} 6x + y\, dA$ where D is the bounded region between $y = x^2$ and $y = 4$.

In problems 21-24 determine the average value of the function f(x,y) on the given domain.

21. $f(x,y) = 3 + 4x + 2y$ where $D = \{(x,y): 0 \le x \le 2 \text{ and } 0 \le y \le 2x\}$

22. $f(x,y) = 1 + 2x + 3y$ where $D = \{(x,y): -2 \le x \le 2 \text{ and } 0 \le y \le 4 - x^2\}$

23. $f(x,y) = x + y^2$ where $D = \{(x,y): 0 \le x \le 3 \text{ and } 0 \le y \le 3 - x\}$

24. $f(x,y) = 2 + xy$ where $D = \{(x,y): 0 \le x \le 2 \text{ and } x^2 \le y \le 6 - x\}$

In problems 25 – 36, change the order of integration. (It helps to sketch the region.)

25. $\int_{0}^{1} \int_{0}^{x} f(x,y)\ dy\ dx$

26. $\int_{-2}^{2} \int_{x^2}^{4} f(x,y)\ dy\ dx$

27. $\int_{0}^{6} \int_{0}^{3-y/2} f(x,y)\ dx\ dy$

28. $\int_{0}^{4} \int_{0}^{\sqrt{x}} f(x,y)\ dy\ dx$

29. $\int_{1}^{2} \int_{0}^{\ln(x)} f(x,y)\ dy\ dx$

30. $\int_{0}^{4} \int_{y/2}^{2} f(x,y)\ dx\ dy$

31. $\int_{0}^{2} \int_{0}^{4-2x} f\ dy\ dx$

32. $\int_{0}^{3} \int_{0}^{9-x^2} f\ dy\ dx$

33. $\int_{0}^{4} \int_{\sqrt{x}}^{2} f\ dy\ dx$

34. $\int_{1}^{3} \int_{2}^{2y} f\ dx\ dy$

35. $\int_{0}^{1} \int_{1}^{e^x} f\ dy\ dx$

36. $\int_{0}^{4} \int_{-\sqrt{y}}^{\sqrt{y}} f\ dx\ dy$

In problems 37 and 38 the depths (in meters) of a small pond are shown. Give a good estimate of the volume of water in the pond. You should be able to justify why your estimate is reasonable.
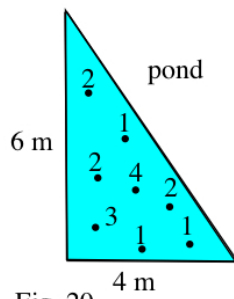
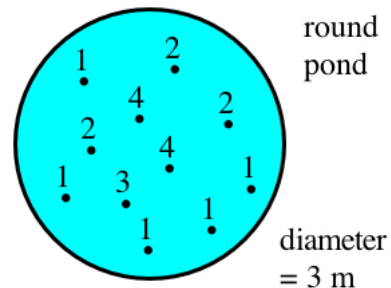37. Fig. 20   38. Fig. 21

Fig. 20

Fig. 21

PRACTICE ANSWERS

Practice 1: $\iint\limits_{D} (2 + x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} 2 + x + 2y)\, dy\, dx$

Starting with the inside integral $\int_{2x^2}^{1+x^2} 2 + x + 2y \ dy = 2y + xy + y^2 \ \Big|_{2x^2}^{1+x^2}$

$= \{2(1+x^2) + x(1+x^2) + (1+x^2)^2\} - \{2(2x^2) + x(2x^2) + (2x^2)^2\} = -3x^4 - x^3 + x + 3$

Then $\int_{-1}^{1} -3x^4 - x^3 + x + 3 \ dx = = -\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{1}{2}x + 3x \ \Big|_{-1}^{1} = \left(\frac{53}{20}\right) - \left(\frac{-43}{20}\right) = \frac{24}{5}$

The Maple command int(f(x,y),y=2*x^2..1+x^2,x=-1..1); gives 24/5.

Practice 2: See Fig. 22. The outside integral (corresponding to the dx) goes
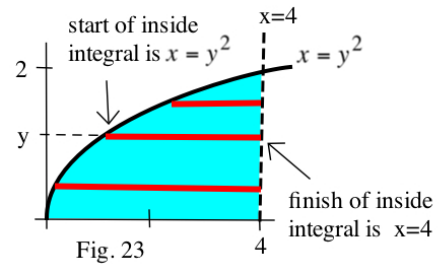from x=0 to x=2. The inside integral begins at y=0
and ends at y=2x.

$\iint\limits_{D} f(x,y)\, dy\, dx = \int_{x=0}^{x=2} \int_{y=0}^{y=2x} f(x,y) \ dy \ dx$

Fig. 22
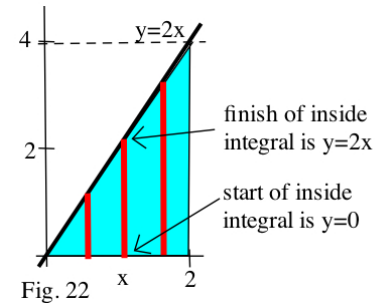
Practice 3: See Fig. 23. The outside integral goes from y=0 to y=2.

$y = \sqrt{x}$ becomes $x = y^2$. Then for each y value

between 0 and 2, the horizontal slice enters the domain

at $x = y^2$ and exits at x=4.

$\iint\limits_{D} f(x,y)\, dx\, dy = \int_{y=0}^{y=2} \int_{x=y^2}^{x=4} f(x,y) \ dx \ dy$

Fig. 23

Practice 4: Since the dy is on the outside, we need to partition the 0≤y≤4 interval. Then

$\iint\limits_{D} f(x,y)\, dx\, dy = \int_{y=0}^{y=2} \int_{x=0}^{x=\sqrt{y}} f(x,y) \ dx \ dy + \int_{y=2}^{y=4} \int_{x=\sqrt{y}}^{x=2} f(x,y) \ dx \ dy$

Practice 5: Version (b) is easier sine we can evaluate $\int_{x=0}^{y} \cos(y^2) \ dx = x \cdot \cos(y^2) \ \Big|_{x=0}^{x=y} = y \cdot \cos(y^2)$

and we cannot evaluate $\int_{y=x}^{1} \cos(y^2) \ dy$.