8.6 Integrals of Trigonometric Functions

There are an overwhelming number of combinations of trigonometric functions which appear in integrals, but fortunately they fall into a few patterns and most of their integrals can be found using reduction formulas and tables of integrals. This section examines some of the patterns of these combinations and illustrates how some of their integrals can be derived.

Products of Sine and Cosine: $\int \sin(ax)\cdot\sin(bx)\,dx$, $\int \cos(ax)\cdot\cos(bx)\,dx$, $\int \sin(ax)\cdot\cos(bx)\,dx$

All of these integrals are handled by referring to the trigonometric identities for sine and cosine of sums and differences:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$$
$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

By adding or subtracting the appropriate pairs of identities, we can write the various products such as $\sin(ax)\cos(bx)$ as a sum or difference of single sines or cosines. For example, by adding the first two identities we get $2\sin(A)\cos(B) = \sin(A + B) + \sin(A - B)$ so $\sin(A)\cos(B) = \frac{1}{2}\{ \sin(A+B) + \sin(A-B) \}$. Using this last identity, the integral of $\sin(ax)\cos(bx)$ for $a \neq b$ is relatively easy:

$$\int \sin(ax)\cos(bx)\,dx = \int \frac{1}{2}\{ \sin((a+b)x) + \sin((a-b)x) \}\,dx = \frac{1}{2}\left\{ \frac{-\cos((a-b)x)}{a-b} + \frac{-\cos((a+b)x)}{a+b} \right\} + C.$$

The other integrals of products of sine and cosine follow in a similar manner.

If $a \neq b$, then

$$\int \sin(ax)\cdot\sin(bx)\,dx = \frac{1}{2}\left\{ \frac{\sin((a-b)x)}{a-b} - \frac{\sin((a+b)x)}{a+b} \right\} + C$$

$$\int \cos(ax)\cdot\cos(bx)\,dx = \frac{1}{2}\left\{ \frac{\sin((a-b)x)}{a-b} + \frac{\sin((a+b)x)}{a+b} \right\} + C$$

$$\int \sin(ax)\cdot\cos(bx)\,dx = \frac{-1}{2}\left\{ \frac{\cos((a-b)x)}{a-b} + \frac{\cos((a+b)x)}{a+b} \right\} + C$$

If $a = b$, we have patterns we have already used.

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} + C = \frac{x}{2} - \frac{\sin(ax)\cdot\cos(ax)}{2a} + C$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} + C = \frac{x}{2} + \frac{\sin(ax)\cdot\cos(ax)}{2a} + C$$

$$\int \sin(ax)\cdot\cos(ax)\,dx = \frac{\sin^2(ax)}{2a} + C = \frac{1-\cos(2ax)}{4a} + C$$

The first and second of these integral formulas follow from the identities $\sin^2(ax) = \dfrac{1-\cos(2ax)}{2}$ and $\cos^2(ax) = \dfrac{1+\cos(2ax)}{2}$, and the third can be derived by changing the variable to $u = \sin(ax)$.

Powers of Sine and Cosine Alone: $\int \sin^n(x)\,dx$, $\int \cos^n(x)\,dx$

All of these antiderivatives can be found using integration by parts or the reduction formulas (formulas 19 and 20 in the integral tables) which were derived using integration by parts. For small values of m and n it is just as easy to find the antiderivatives directly.

Even Powers of Sine or Cosine Alone

For **even** powers of sine or cosine, we can successfully reduce the size of the exponent by repeatedly applying the identities $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$ and $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$.

Example 1: Evaluate $\int \sin^4(x)\,dx$.

Solution: $\sin^4(x) = \{ \sin^2(x) \}^2 = \{ \frac{1}{2} [1 - \cos(2x)] \}^2 = \frac{1}{4} \{ 1 - 2\cos(2x) + \cos^2(2x) \}$ so

$$\int \sin^4(x)\,dx = \int \frac{1}{4} \{ 1 - 2\cos(2x) + \cos^2(2x) \}\,dx$$

$$= \frac{1}{4} \left\{ x + \sin(2x) + \frac{x}{2} + \frac{\sin(2x)\cos(2x)}{2} \right\} + C .$$

Practice 1: Evaluate $\int \cos^4(x)\,dx$.

Odd Powers of Sine or Cosine Alone

For odd powers of sine or cosine we can split off one factor of sine or cosine, reduce the remaining even exponent using the identities $\sin^2(x) = 1 - \cos^2(x)$ or $\cos^2(x) = 1 - \sin^2(x)$, and finally integrate by changing the variable.

Example 2: Evaluate $\int \sin^5(x)\,dx$.

Solution: $\sin^5(x) = \sin^4(x)\sin(x) = \{\sin^2(x)\}^2 \sin(x) = \{1 - \cos^2(x)\}^2 \sin(x)$
$= \{1 - 2\cos^2(x) + \cos^4(x)\}\sin(x)$.

Then $\int \sin^5(x)\,dx = \int \sin(x)\,dx - 2\int \cos^2(x)\sin(x)\,dx + \int \cos^4(x)\sin(x)\,dx$.

The first integral is easy, and the last two can be evaluated by changing the variable to $u = \cos(x)$:

$$\int \sin^5(x)\,dx = -\cos(x) - 2\left\{-\frac{\cos^3(x)}{3}\right\} + \left\{-\frac{\cos^5(x)}{5}\right\} + C.$$

Practice 2: Evaluate $\int \cos^5(x)\,dx$.

Patterns for $\int \sin^m(x)\cos^n(x)\,dx$

If the exponent of sine is odd, we can split off one factor $\sin(x)$ and use the identity $\sin^2(x) = 1 - \cos^2(x)$ to rewrite the remaining even power of sine in terms of cosine. Then the change of variable $u = \cos(x)$ makes all of the integrals straightforward.

Example 3: Evaluate $\int \sin^3(x)\cos^6(x)\,dx$.

Solution: $\sin^3(x)\cos^6(x) = \sin(x)\,\mathbf{\sin^2(x)}\cos^6(x) = \sin(x)\,\{\mathbf{1 - \cos^2(x)}\}\cos^6(x)$

$= \sin(x)\cos^6(x) - \sin(x)\cos^8(x)$.

Then $\int \sin^3(x)\cos^6(x)\,dx = \int \sin(x)\cos^6(x) - \sin(x)\cos^8(x)\,dx$ (put $u = \cos(x)$)

$$= -\frac{\cos^7(x)}{7} + \frac{\cos^9(x)}{9} + C.$$

Practice 3: Evaluate $\int \sin^3(x)\cos^4(x)\,dx$.

If the **exponent of cosine is odd**, we can split off one factor $\cos(x)$ and use the identity $\cos^2(x) = 1 - \sin^2(x)$ to rewrite the remaining even power of cosine in terms of sine. Then the change of variable $u = \sin(x)$ makes all of the integrals straightforward.

If **both exponents are even**, we can use the identities $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ to rewrite the integral in terms of powers of $\cos(2x)$ and then proceed with integrating even powers of cosine.

Powers of Secant and Tangent Alone: $\int \sec^n(x)\,dx$, $\int \tan^n(x)\,dx$

All of the integrals of powers of secant and tangent can be evaluated by knowing

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C \quad \text{and}$$

$$\int \tan(x)\,dx = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$$

and then using the reduction formulas

$$\int \sec^n(x)\,dx = \frac{\sec^{n-2}(x)\cdot\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx \quad \text{and}$$

$$\int \tan^n(x)\,dx = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,dx\ .$$

Example 4: Evaluate $\int \sec^3(x)\,dx$.

Solution: Using the reduction formula with $n = 3$,

$$\int \sec^3(x)\,dx = \frac{\sec(x)\cdot\tan(x)}{2} + \frac{1}{2}\int \sec(x)\,dx = \frac{\sec(x)\cdot\tan(x)}{2} + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C.$$

Practice 4: Evaluate $\int \tan^3(x)\,dx$ and $\int \sec^5(x)\,dx$.

Patterns for $\int \sec^m(x)\cdot\tan^n(x)\,dx$

The patterns for evaluating $\int \sec^m(x)\cdot\tan^n(x)\,dx$ are similar to those for $\int \sin^m(x)\cdot\cos^n(x)\,dx$ because we treat the even and odd powers differently and we use the identities $\tan^2(x) = \sec^2(x) - 1$ and $\sec^2(x) = \tan^2(x) + 1$.

If the **exponent of secant is even**, factor off $\sec^2(x)$, replace the other even powers (if any) of secant using $\sec^2(x) = \tan^2(x) + 1$, and make the change of variable $u = \tan(x)$ (then $du = \sec^2(x)\,dx$).

If the **exponent of tangent is odd**, factor off $\sec(x)\tan(x)$, replace the remaining even powers (if any) of tangent using $\tan^2(x) = \sec^2(x) - 1$, and make the change of variable $u = \sec(x)$ (then $du = \sec(x)\tan(x)\,dx$).

If the **exponent of secant is odd and the exponent of tangent is even**, replace the even powers of tangent using $\tan^2(x) = \sec^2(x) - 1$. Then the integral contains only powers of secant, and we can use the patterns for integrating powers of secant alone.

Example 5: Evaluate $\int \sec(x)\cdot\tan^2(x)\,dx$.

Solution: Since the exponent of secant is odd and and the exponent of tangent is even, we can use the last method mentions: replace the even powers of tangent using $\tan^2(x) = \sec^2(x) - 1$. Then

$$\int \sec(x)\cdot\tan^2(x)\,dx = \int \sec(x)\cdot\{\sec^2(x) - 1\}\,dx$$

$$= \int \sec^3(x) - \sec(x)\,dx = \int \sec^3(x)\,dx - \int \sec(x)\,dx$$

$$= \left\{ \frac{\sec(x)\cdot\tan(x)}{2} + \frac{1}{2}\ln|\sec(x) + \tan(x)| \right\} - \ln|\sec(x) + \tan(x)| + C$$

$$= \frac{\sec(x)\cdot\tan(x)}{2} - \frac{1}{2}\ln|\sec(x) + \tan(x)| + C.$$

Practice 5: Evaluate $\int \sec^4(x)\cdot\tan^2(x)\,dx$.

Wrap Up

Even if you use tables of integrals (or computers) for most of your future work, it is important to realize that most of the integral formulas can be derived from some basic facts using the techniques we have discussed in this and earlier sections.

PROBLEMS

Evaluate the integrals. (More than one method works for some of the integrals.)

1. $\int \sin^2(3x)\,dx$
2. $\int \cos^2(5x)\,dx$
3. $\int e^x\cdot\sin(e^x)\cdot\cos(e^x)\,dx$
4. $\int \frac{1}{x}\cdot\sin^2(\ln(x))\,dx$
5. $\int_0^{\pi} \sin^4(3x)\,dx$
6. $\int_0^{\pi} \cos^4(5x)\,dx$

7. $\int_0^{\pi} \sin^3(7x)\ dx$

8. $\int_0^{\pi} \cos^3(5x)\ dx$

9. $\int \sin(7x)\cdot\cos(7x)\ dx$

10. $\int \sin(7x)\cdot\cos^2(7x)\ dx$

11. $\int \sin(7x)\cdot\cos^3(7x)\ dx$

12. $\int \sin^2(3x)\cdot\cos(3x)\ dx$

13. $\int \sin^2(3x)\cdot\cos^2(3x)\ dx$

14. $\int \sin^2(3x)\cdot\cos^3(3x)\ dx$

15. $\int \sec^2(5x)\cdot\tan(5x)\ dx$

16. $\int \sec^2(3x)\cdot\tan^2(3x)\ dx$

17. $\int \sec^3(3x)\cdot\tan(3x)\ dx$

18. $\int \sec^3(5x)\cdot\tan^2(5x)\ dx$

The definite integrals of various combinations of sine and cosine on the interval $[0, 2\pi]$ exhibit a number of interesting patterrns. For now these patterns are simply curiousities and a source of additional problems for practice, but the patterns are very important as the foundation for an applied topic, Fourier Series, that you may encounter in more advanced courses.

The next three problems ask you to show that the definite integral on $[0, 2\pi]$ of $\sin(mx)$ multiplied by almost any other combination of $\sin(nx)$ or $\cos(nx)$ is 0. The only nonzero value comes when $\sin(mx)$ is multiplied by itself.

19. Show that if m and n are integers with $m \neq n$, then $\int_0^{2\pi} \sin(mx)\cdot\sin(nx)\ dx = 0$.

20. Show that if m and n are integers, then $\int_0^{2\pi} \sin(mx)\cdot\cos(nx)\ dx = 0$. (Consider $m = n$ and $m \neq n$.)

21. Show that if $m \neq 0$ is an integer, then $\int_0^{2\pi} \sin(mx)\cdot\sin(mx)\ dx = \pi$.

22. Suppose $P(x) = 5\cdot\sin(x) + 7\cdot\cos(x) - 4\cdot\sin(2x) + 8\cdot\cos(2x) - 2\cdot\sin(3x)$. (This is called a trigonometric polynomial.) Use the **results** of problems 19–21 to quickly evaluate

(a) $a_1 = \frac{1}{\pi}\int_0^{2\pi} \sin(1x)\cdot P(x)\ dx$

(b) $a_2 = \frac{1}{\pi}\int_0^{2\pi} \sin(2x)\cdot P(x)\ dx$

(c) $a_3 = \frac{1}{\pi}\int_0^{2\pi} \sin(3x)\cdot P(x)\ dx$

(d) $a_4 = \frac{1}{\pi}\int_0^{2\pi} \sin(4x)\cdot P(x)\ dx$

(e) Describe how the values of a_i are related to the coeffiecients of $P(x)$.

(f) Make up your own trigonometric polynomial $P(x)$ and see if your description in part (e) holds for the a_i values calculated from the new $P(x)$.

(g) Just by knowing the a_i values we can "rebuild" part of $P(x)$. Find a similar method for getting the coefficients of the cosine terms of $P(x)$: $b_i = ??$

23. Show that if n is a positive, **odd** integer, then $\int_0^{2\pi} \sin^n(x)\ dx = 0$.

24. It is straightforward (using formula 19 in the integral table) to show that $\int_0^{2\pi} \sin^2(x)\ dx = \pi$, $\int_0^{2\pi} \sin^4(x)\ dx = \frac{3}{4}\pi$, and $\int_0^{2\pi} \sin^6(x)\ dx = \frac{5}{6}\frac{3}{4}\pi$. (a) Evaluate $\int_0^{2\pi} \sin^8(x)\ dx$.

(b) Predict the value of $\int_0^{2\pi} \sin^{10}(x)\ dx$ and then evalaute the integral.

Section 8.6 Practice Answers

Practice 1: $\int \cos^4(x)\,dx$ { Use $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ }

$$= \int \cos^2(x)\cdot\cos^2(x)\,dx = \int \frac{1}{2}(1 + \cos(2x))\frac{1}{2}(1 + \cos(2x))\ dx$$

$$= \frac{1}{4}\int 1 + 2\cos(2x) + \cos^2(2x)\ dx = \frac{1}{4}\int 1 + 2\cos(2x) + \frac{1}{2}\{1 + \cos(4x)\}\ dx$$

$$= \frac{1}{4}\int \frac{3}{2} + 2\cos(2x) + \frac{1}{2}\cos(4x)\ dx = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C\ .$$

Practice 2: $\int \cos^5(x)\,dx = \int \cos^2(x)\cdot\cos^2(x)\cdot\cos(x)\,dx = \int (1 - \sin^2(x))(1 - \sin^2(x))\cos(x)\,dx$

$$= \int \{1 - 2\sin^2(x) + \sin^4(x)\}\cos(x)\ dx$$

$$= \int \cos(x)\,dx - 2\int \sin^2(x)\cdot\cos(x)\,dx + \int \sin^4(x)\cdot\cos(x)\,dx \quad (\text{Use } u = \sin(x),\ du = \cos(x)\,dx)$$

$$= \sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C\ .$$

Practice 3: $\int \sin^3(x)\cdot\cos^4(x)\,dx = \int \sin(x)\cdot\sin^2(x)\cdot\cos^4(x)\,dx = \int \sin(x)\cdot(1 - \cos^2(x))\cdot\cos^4(x)\,dx$

$$= \int \sin(x)\cdot\cos^4(x)\,dx - \int \sin(x)\cdot\cos^6(x)\,dx \quad (\text{Use } u = \cos(x),\ du = -\sin(x)\,dx)$$

$$= -\frac{1}{5}\cos^5(x) + \frac{1}{7}\cos^7(x) + C$$

Practice 4: $\int \tan^3(x)\,dx = \frac{1}{2}\tan^2(x) - \int \tan(x)\,dx = \frac{1}{2}\tan^2(x) - \ln|\sec(x)| + C$.

$$\int \sec^5(x)\,dx = \frac{1}{2}\sec^3(x)\cdot\tan(x) + \frac{3}{4}\int \sec^3(x)\,dx$$

$$= \frac{1}{2}\sec^3(x)\cdot\tan(x) + \frac{3}{4}\left\{\frac{1}{2}\sec(x)\cdot\tan(x) + \frac{1}{2}\int \sec(x)\,dx\right\}$$

$$= \frac{1}{2}\sec^3(x)\cdot\tan(x) + \frac{3}{8}\sec(x)\cdot\tan(x) + \frac{3}{8}\ln|\sec(x)+\tan(x)| + C.$$

Practice 5: $\int \sec^4(x)\cdot\tan^2(x)\,dx = \int \sec^2(x)\cdot\sec^2(x)\cdot\tan^2(x)\,dx$

$$= \int \sec^2(x)\cdot(\tan^2(x)+1)\cdot\tan^2(x)\,dx$$

$$= \int \sec^2(x)\cdot\tan^4(x)\,dx + \int \sec^2(x)\cdot\tan^2(x)\,dx \quad (\text{Use } u = \tan(x),\ du = \sec^2(x)\,dx)$$

$$= \frac{1}{5}\tan^5(x) + \frac{1}{3}\tan^3(x) + C.$$