4.4 AREAS, INTEGRALS AND ANTIDERIVATIVES

This section explores properties of functions defined as areas and examines some of the connections among areas, integrals and antiderivatives. In order to focus on the geometric meaning and connections, all of the functions in this section are nonnegative, but the results are generalized in the next section and proved true for all continuous functions. This section also introduces examples to illustrate how areas, integrals and antiderivatives can be used.

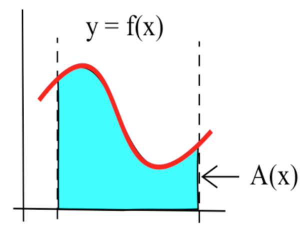

Fig. 1

When f is a continuous, **nonnegative** function, then the "area function" $A(x) = \int_a^x f(t)\,dt$ represents the area between the graph of f, the t–axis, and between the vertical lines at t=a and t=x (Fig. 1), and the derivative of $A(x)$ represents the rate of change (growth) of $A(x)$. Examples 2 and 3 of Section 4.3 showed that for some functions f, the derivative of A(x) was equal to f so A(x) was an antiderivative of f. The next theorem says the result is true for every continuous, nonnegative function f.

The Area Function is an Antiderivative

If f is a continuous nonnegative function, x ≥ a, and $A(x) = \int_a^x f(t)\,dt$

then $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \frac{d}{dx} A(x) = f(x)$, so $A(x)$ is an antiderivative of $f(x)$.

This result relating integrals and antiderivatives is a special case (for nonnegative functions f) of the Fundamental Theorem of Calculus (Part 1) which is proved in Section 4.5 . This result is important for two reasons:

(i) it says that a large collection of functions have antiderivatives, and

(ii) it leads to an **easy** way of **exactly** evaluating definite integrals.

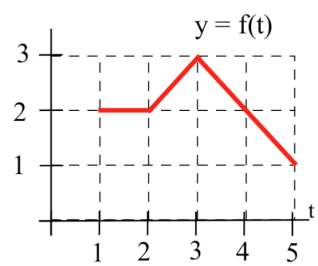

Fig. 2

Example 1: $A(x) = \int_1^x f(t)\,dt$ for the function $f(t)$ shown in Fig. 2.

Estimate the values of $A(x)$ and $A'(x)$ for x = 2, 3, 4 and 5 and use these values to sketch the graph of $y = A(x)$.

Solution: Dividing the region into squares and triangles, it is easy to see that A(2) = 2, A(3) = 4.5, A(4) = 7 , and A(5) = 8.5 . Since $A'(x) = f(x)$, we know that A'(2) = f(2) = 2 , A'(3) = f(3) = 3 , A'(4) = f(4) = 2 ,

and A'(5) = f(5) = 1. The graph of $y = A(x)$ is shown in Fig. 3.

It is important to recognize that f is not differentiable at $x = 2$ and $x = 3$. However, the values of A change smoothly near 2 and 3, and the function A is differentiable at those points and at every other point from 1 to 5. Also, $f'(4) = -1$ (f is clearly decreasing near $x = 4$), but $A'(4) = f(4) = 2$ is positive (the area A is growing even though f is getting smaller).

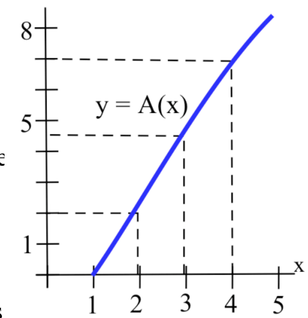

Fig. 3

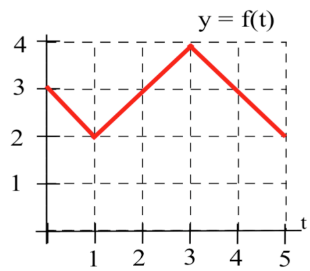

Fig. 4

Practice 1: B(x) is the area bounded by the horizontal axis, vertical lines at $t = 0$ and $t = x$, and the graph of f(t) shown in Fig. 4. Estimate the values of B(x) and B'(x) for $x = 1, 2, 3, 4$ and 5.

Example 2: Let $G(x) = \frac{d}{dx}\left(\int_0^x \sin(t)\, dt \right)$. Evaluate G($x$) for $x = \pi/4, \pi/2$, and $3\pi/4$.

Solution: Fig. 5(a) shows the graph of $A(x) = \int_0^x \sin(t)\, dt$, and G(x) is the derivative of A(x). By the theorem, $A'(x) = \sin(x)$ so $A'(\pi/4) = \sin(\pi/4) \approx .707$, $A'(\pi/2) = \sin(\pi/2) = 1$, and $A'(3\pi/4) = \sin(3\pi/4) \approx .707$. Fig. 5(b) shows the graph of $y = A(x)$ and 5(c) is the graph of $y = A'(x) = G(x)$

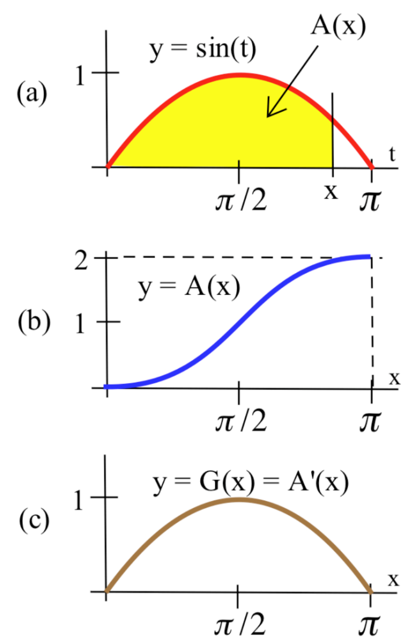

Fig. 5

Using Antiderivatives to Evaluate $\int_a^b f(x)\, dx$

Now we can put the ideas of areas and antiderivatives together to get a way of evaluating definite integrals that is exact and often easy.

If $A(x) = \int_a^x f(t)\, dt$, then $A(a) = \int_a^a f(t)\, dt = 0$, $A(b) = \int_a^b f(t)\, dt$, and A(x) is an antiderivative of f, $A'(x) = f(x)$. We also know that if F(x) is **any** antiderivative of f, then F(x) and A(x) have the same derivative so F(x) and A(x) are "parallel" and differ by a constant, $F(x) = A(x) + C$ for all x.

Then $F(b) - F(a) = \{ A(b) + C \} - \{ A(a) + C \} = A(b) - A(a) = \int_a^b f(t)\, dt - \int_a^a f(t)\, dt = \int_a^b f(t)\, dt$.

To evaluate a definite integral $\int_a^b f(t)\,dt$, we can find any antiderivative F of f and evaluate F(b) – F(a).

This result is a special case of Part 2 of the Fundamental Theorem of Calculus, and it will be used hundreds of times in the next several chapters. The Fundamental Theorem is stated and proved in Section 4.5.

Antiderivatives and Definite Integrals

If f is a continuous, nonnegative function

and **F is any antiderivative** of f ($F'(x) = f(x)$) on the interval [a,b] ,

then $\left\{ \begin{array}{l} \text{area bounded between the graph} \\ \text{of f and the } x\text{–axis and} \\ \text{vertical lines at } x = a \text{ and } x = b \end{array} \right\} = \boldsymbol{\int_a^b f(x)\,dx = F(b) - F(a)}$.

The problem of finding the exact value of a definite integral reduces to finding some (any) antiderivative F of the integrand and then evaluating F(b) – F(a). Even finding one antiderivative can be difficult, and, for now, we will stick to functions which have easy antiderivatives. Later we will explore some methods for finding antiderivatives of more difficult functions.

The evaluation F(b) – F(a) is represented by the symbol $F(x) \Big|_a^b$.

Example 3: Evaluate $\int_1^3 x\,dx$ in two ways:

(i) By sketching the graph of $y = x$ and geometrically finding the area.

(ii) By finding an antiderivative of F(x) of f and evaluating F(3)–F(1).

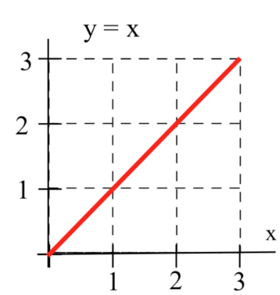

Fig. 6

Solution: (i) The graph of $y = x$ is shown in Fig. 6, and the shaded region has area 4.

(ii) One antiderivative of x is $F(x) = \frac{1}{2}x^2$ (check that $D(x^2/2) = x$), and

$$F(x)\Big|_1^3 = F(3)–F(1) = \frac{1}{2}(3^2) - \frac{1}{2}(1^2) = \frac{9}{2} - \frac{1}{2} = 4 \text{ which agrees with (i).}$$

If someone chose another antiderivative of x, say $F(x) = \frac{1}{2}x^2 + 7$ (check that $\mathbf{D}(x^2/2 + 7) = x$),

then $F(x)\Big|_1^3 = F(3) - F(1) = \left\{ \frac{1}{2}(3^2) + 7 \right\} - \left\{ \frac{1}{2}(1^2) + 7 \right\} = \frac{23}{2} - \frac{15}{2} = 4$.

No matter which antiderivative F is chosen, F(3) – F(1) equals 4.

Practice 2: Evaluate $\int_1^3 (x - 1)\,dx$ in the two ways of the previous example.

This antiderivative method is an extremely powerful way to evaluate some definite integrals, and it is used often. However, it can only be used to evaluate a definite integral of a function defined by a formula.

Example 4: Find the area between the graph of the cosine and the horizontal axis for x between 0 and $\pi/2$.

Solution: The area we want (Fig. 7) is $\int_0^{\pi/2} \cos(x)\,dx$ so we need an antiderivative of $f(x) = \cos(x)$.

$F(x) = \sin(x)$ is one antiderivative of $\cos(x)$ (Check that $\mathbf{D}(\sin(x)) = \cos(x)$). Then

$$\text{area} = \int_0^{\pi/2} \cos(x)\,dx = \sin(x)\Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1.$$

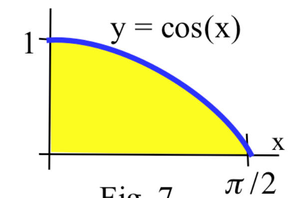

Fig. 7

Practice 3: Find the area between the graph of $y = 3x^2$ and the horizontal axis for x between 1 and 2.

Integrals, Antiderivatives, and Applications

The antiderivative method of evaluating definite integrals can also be used when we need to find an "area", and it is useful for solving applied problems.

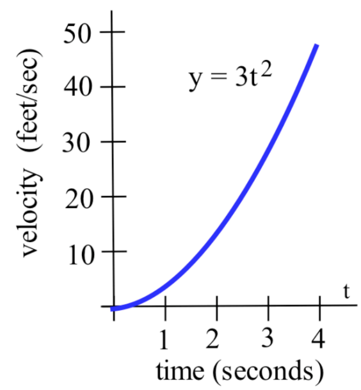

Fig. 8

Example 5: A robot has been programmed so that when it starts to move, its velocity after t seconds will be $3t^2$ feet/second.

(a) How far will the robot travel during its first 4 seconds of movement?

(b) How far will the robot travel during its next 4 seconds of movement?

(c) How many seconds before the robot is 729 feet from its starting place?

Solution: (a) The distance during the first 4 seconds will be the area under the graph (Fig. 8) of velocity , $f(t) = t$, from $t = 0$ to $t = 4$, and that area is the definite integral

$$\int_0^4 3t^2\,dt \quad . \text{ An antiderivative of } 3t^2 \text{ is } t^3 \text{ so } \int_0^4 3t^2\,dt = t^3\Big|_0^4 = (4)^3 - (0)^3 = 64 \text{ feet.}$$

(b) $\displaystyle\int_4^8 3t^2\,dt = t^3\Big|_4^8 = (8)^3 - (4)^3 = 512 - 64 = 448$ feet.

(c) This part is different from the other two parts. Here we are told the lower integration endpoint, $t = 0$, and the total distance, 729 feet, and we are asked to find the upper endpoint. Calling the upper endpoint T, we know that

$$729 = \int_0^T 3t^2\,dt = t^3\Big|_0^T = (T)^3 - (0)^3 = T^3, \quad \text{so } T = \sqrt[3]{729} = 9 \text{ seconds.}$$

Practice 4: (a) How far will the robot move between $t = 1$ second and $t = 5$ seconds?

(b) How many seconds before the robot is 343 feet from its starting place?

Example 6: Suppose that t minutes after putting 1000 bacteria on a Petri plate the rate of growth of the population is $6t$ bacteria per minute. (a) How many new bacteria are added to the population during the first 7 minutes? (b) What is the total population after 7 minutes? (c) When will the total population be 2200 bacteria?

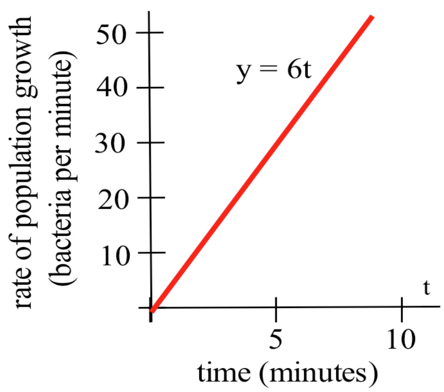

Fig. 9

Solution: (a) The number of new bacteria is the area under the rate of growth graph (Fig. 9), and one antiderivative of $6t$ is $3t^2$ (check that $D(3t^2) = 6t$) so

$$\text{new bacteria} = \int_0^7 6t\,dt = 3t^2 \Big|_0^7 = 3(7)^2 - 3(0)^2 = 147 .$$

(b) The new population = {old population} + {new bacteria} = 1000 + 147 = 1147 bacteria.

(c) If the total population is 2200 bacteria, then there are 2200 – 1000 = 1200 new bacteria, and we need to find the time T needed for that many new bacteria to occur.

$$1200 \text{ new bacteria} = \int_0^T 6t\,dt = 3t^2 \Big|_0^T = 3(T)^2 - 3(0)^2 = 3\,T^2 \quad \text{so} \quad T^2 = 400 \text{ and}$$

T = 20 minutes. After 20 minutes, the total bacteria population will be 1000 + 1200 = 2200.

Practice 5: (a) How many new bacteria will be added to the population between $t = 4$ and $t = 8$ minutes?

(b) When will the total population be 2875 bacteria? (Hint: How many are new?)

PROBLEMS

In problems 1 – 4, the function f is given by a graph, and $A(x) = \int_1^x f(t)\,dt$.

(a) Graph $y = A(x)$ for $1 \le x \le 5$. (b) Estimate the values of $A(1)$, $A(2)$, $A(3)$, and $A(4)$.

(c) Estimate the values of $A'(1)$, $A'(2)$, $A'(3)$, and $A'(4)$.

1. f in Fig. 11. 2. f in Fig. 12. 3. f in Fig. 13 4. f in Fig. 14.

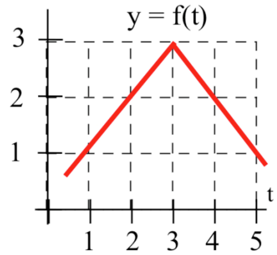

Fig. 11

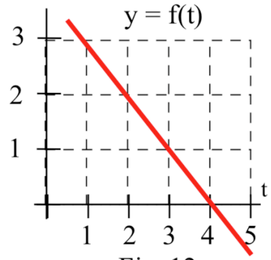

Fig. 12

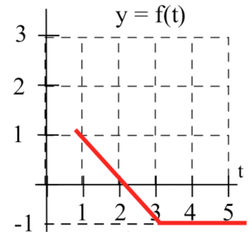

Fig. 13

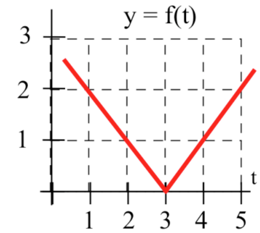

Fig. 14

In problems 5 – 8, the function f is given by a formula, and $A(x) = \int_1^x f(t)\,dt$.

(a) Graph $y = A(x)$ for $1 \le x \le 5$. (b) Calculate the values of $A(1)$, $A(2)$, $A(3)$, and $A(4)$.

(c) Determine the values of $A'(1)$, $A'(2)$, $A'(3)$, and $A'(4)$.

5. $f(t) = 2$ 6. $f(t) = 1 + t$ 7. $f(t) = 6 - t$ 8. $f(t) = 1 + 2t$

In problems 9 – 18, use the **Antiderivatives and Definite Integrals** Theorem to evaluate the integrals.

9. $\int_0^3 2x\,dx$, $\int_1^3 2x\,dx$, $\int_0^1 2x\,dx$

10. $\int_0^2 4x^3\,dx$, $\int_0^1 4x^3\,dx$, $\int_1^2 4x^3\,dx$

11. $\int_1^3 6x^2\,dx$, $\int_1^2 6x^2\,dx$, $\int_0^3 6x^2\,dx$

12. $\int_{-2}^{2} 2x\,dx$, $\int_{-2}^{-1} 2x\,dx$, $\int_{-2}^{0} 2x\,dx$

13. $\int_0^3 4x^3\,dx$, $\int_1^3 4x^3\,dx$, $\int_0^1 4x^3\,dx$

14. $\int_0^5 4x^3\,dx$, $\int_0^2 4x^3\,dx$, $\int_2^5 4x^3\,dx$

15. $\int_{-3}^{3} 3x^2\,dx$, $\int_{-3}^{0} 3x^2\,dx$, $\int_0^3 3x^2\,dx$

16. $\int_0^3 5\,dx$, $\int_0^2 5\,dx$, $\int_2^3 5\,dx$

17. $\int_0^2 3x^2\,dx$, $\int_1^3 3x^2\,dx$

18. $\int_{-2}^{2} 12 - 3x^2 \, dx$, $\int_{0}^{2} 12 - 3x^2 \, dx$, $\int_{1}^{2} 12 - 3x^2 \, dx$

19. The velocity of a car after t seconds is 2t feet per second. (a) How far does the car travel during its first 10 seconds? (b) How many seconds does it take the car to travel half the distance in part (a)?

20. The velocity of a car after t seconds is $3t^2$ feet per second. (a) How far does the car travel during its first 10 seconds? (b) How many seconds does it take the car to travel half the distance in part (a)?

21. The velocity of a car after t seconds is $4t^3$ feet per second. (a) How far does the car travel during its first 10 seconds? (b) How many seconds does it take the car to travel half the distance in part (a)?

22. The velocity of a car after t seconds is $20 - 2t$ feet per second. (a) How many seconds does it take for the car to come to a stop (velocity = 0)? (b) How far does the car travel while coming to a stop? (c) How many seconds does it take the car to travel half the distance in part (b)?

23. The velocity of a car after t seconds is $75 - 3t^2$ feet per second. (a) How many seconds does it take for the car to come to a stop (velocity = 0)? (b) How far does the car travel while coming to a stop? (c) How many seconds does it take the car to travel half the distance in part (b)?

24. Find the exact area under half of one arch of the sine curve: $\int_{0}^{\pi/2} \sin(x) \, dx$. (Note: $\mathbf{D}(-\cos(x)) = \sin(x)$)

25. An artist you know wants to make a figure consisting of the region between the curve $y = x^2$ and the x–axis for $0 \le x \le 3$.
   (a) Where should the artist divide the region with a vertical line (Fig. 15) so each piece has the same area?
   (b) Where should the artist divide the region with vertical lines to get 3 pieces with equal areas?

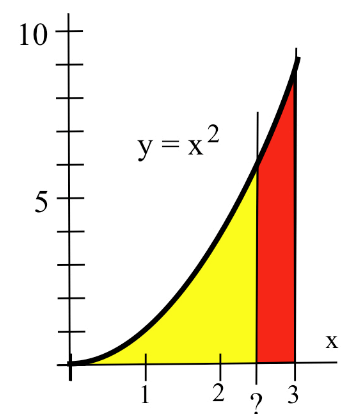

Fig. 15

Section 4.4 **PRACTICE Answers**

Practice 1: B(1) = 2.5, B(2) = 5, B(3) = 8.5, B(4) = 12, B(5) = 14.5

$B(x) = \int_0^x f(t)\,dt$ so $B'(x) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$ (by The Area Function is an Antiderivative theorem): then $B'(1) = f(1) = 2$, $B'(2) = f(2) = 3$, $B'(3) = 4$, $B'(4) = 3$, and $B'(5) = 2$.

Practice 2: (a) As an area, $\int_1^3 x - 1\,dx$ is the area of the triangular region between $y = x - 1$ and the x–axis for $1 \le x \le 3$: area $= \frac{1}{2}$(base)(height) $= \frac{1}{2}(2)(2) = 2$.

(b) $F(x) = \frac{x^2}{2} - x$ is an antiderivative of $f(x) = x - 1$ so

area $= \int_1^3 x - 1\,dx = F(3) - F(1) = \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) = 2.$

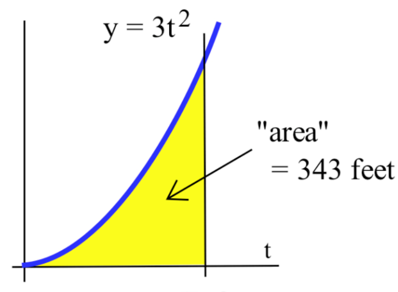

Fig. 16

Practice 3: Area $= \int_1^2 3x^2\,dx = x^3 \Big|_1^2 = 8 - 1 = \mathbf{7}$.

Practice 4: (a) distance $= \int_1^5 3t^2\,dt = t^3 \Big|_1^5 = 125 - 1 =$ **124 feet**.

(b) In this problem we know the starting point is x = 0, and the total distance ("area") is 343 feet. Our problem is to find the time T (Fig. 16) so 343 feet $= \int_0^T 3t^2\,dt$.

$343 = \int_0^T 3t^2\,dt = t^3 \Big|_0^T = T^3 - 0 = T^3$ so T = **7 seconds**.

Practice 5: (a) number of new bacteria $= \int_4^8 6t\,dt$

$= 3t^2 \Big|_4^8 = 3\cdot 64 - 3\cdot 16 = 144$ bacteria.

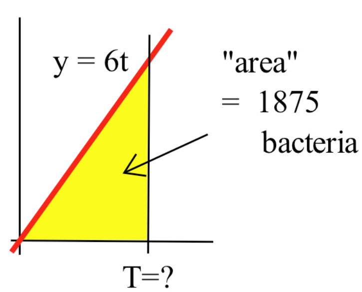

Fig. 17

(b) We know the total new population ("area" in Fig. 17) is $2875 - 1000 = 1875$ so

$1875 = \int_0^T 6t\,dt = 3t^2 \Big|_0^T = 3T^2 - 0 = 3T^2$ so T = **25 minutes**.