$9.4\frac{1}{2}$ BEZIER CURVES — Getting the shape you want

Historically, parametric equations were often used to model the motion of objects, and that is the approach we have seen so far. But more recently, as computers became more common in design work and manufacturing, a need arose to efficiently find formulas for shapes such as airplane wings and automobile bodies and even letters of the alphabet that designers or artists had created.

One simple but inefficient method for describing and storing the shape of a curve is to measure the location and save the coordinates of hundreds or thousands of points along the curve. This result is called a "bitmap" of the shape. However, bitmaps typically require a large amount of computer memory, and when the bitmap is reconverted from stored coordinates back into a graphic image, originally smooth curves often appear jagged (Fig. 1). Also, when these bitmapped shapes are stretched or rotated, the new location of every one of the points must be calculated, a relatively slow process.

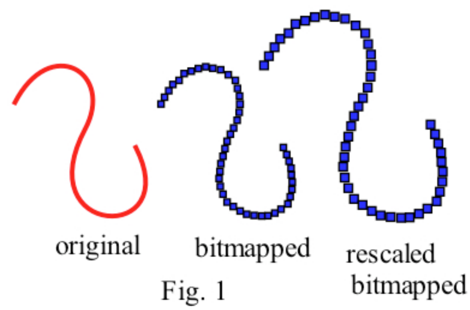

Fig. 1

A second method, still simple but more efficient than bitmaps, is to store fewer points along the curve, but to automatically connect consecutive points with line segments (Fig. 2). Less computer memory is required since fewer coordinates are stored, and stretches and rotations are calculated more quickly since the new locations of fewer points are needed. This method is commonly used in computer graphics to store and redraw surfaces (Fig. 3). Sometimes instead of saving the coordinates of each point, a "vector" is used to describe how to get to the next point from the previous point, and the result is a "vector map" of the curve. The major drawback of this method is that the stored and redrawn curve consists of straight segments and corners even though the original curve may have been smooth.

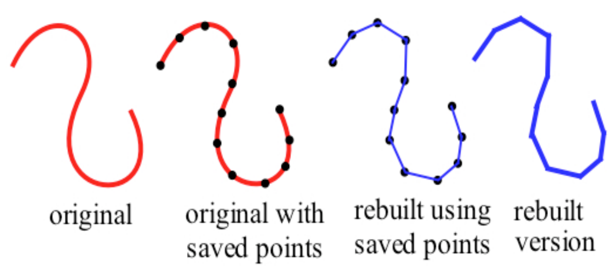

Fig. 2

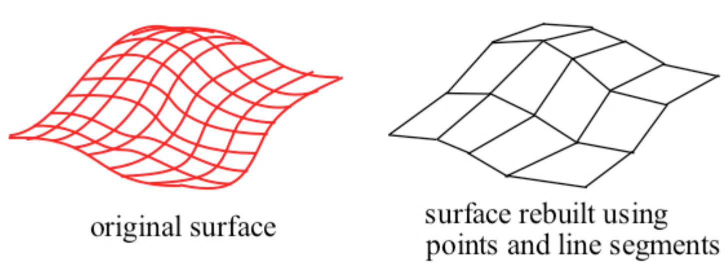

Fig. 3

The primary building block for curves and surfaces represented as line segments is the line segment given by parametric equations.

Example 1: Show that the parametric equations

$x(t) = (1 - t)\cdot x_0 + t\cdot x_1$ and

$y(t) = (1 - t)\cdot y_0 + t\cdot y_1$

for $0 \le t \le 1$ go through the points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ (Fig. 4) and that the slope is

$$\frac{dy}{dx} = \frac{y_1 - y_0}{x_1 - x_0}$$

for all values of t between 0 and 1.

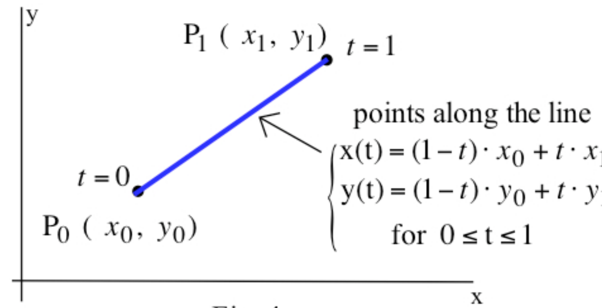

Fig. 4

We typically abbreviate the pair of parametric equations as $P(t) = (1 - t)\cdot P_0 + t\cdot P_1$.

Solution: $x(\mathbf{0}) = (1 - \mathbf{0})\cdot x_0 + \mathbf{0}\cdot x_1 = x_0$ and $y(\mathbf{0}) = (1 - \mathbf{0})\cdot y_0 + \mathbf{0}\cdot y_1 = y_0$ so $P(\mathbf{0}) = P_0$.

$x(\mathbf{1}) = (1 - \mathbf{1})\cdot x_0 + \mathbf{1}\cdot x_1 = x_1$ and $y(\mathbf{1}) = (1 - \mathbf{1})\cdot y_0 + \mathbf{1}\cdot y_1 = y_1$ so $P(\mathbf{1}) = P_1$.

$$\frac{d\,y(t)}{d\,t} = -y_0 + y_1 \quad \text{and} \quad \frac{d\,x(t)}{d\,t} = -x_0 + x_1$$

so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{y_1 - y_0}{x_1 - x_0}$, the slope of the line segment from P_0 to P_1 .

Practice 1: Use the pattern of Example 1 to write parametric equations for the line segments that (a) connect (1,2) to (5, 4), (b) connect (5, 4) to (1, 2), and (c) connect (6, –2) to (3, 1).

Bezier Curves

One solution to the problem of efficiently saving and redrawing a smooth curve was independently developed in the 1960s by two French automobile engineers, Pierre Bezier (pronounced "bez–ee–ay") who worked for Renault automobile company and P. de Casteljau who worked for Citroen. Originally, the solutions were considered industrial secrets, but Bezier's work was eventually published first. The curves that result using Bezier's method are called Bezier curves. The method of Bezier curves allow us to efficiently store information about smooth (and not–so–smooth) shapes and to quickly stretch, rotate and distort these shapes. Bezier curves are now commonly used in computer–aided design work and in most computer drawing programs. They are also used to specify the shapes of letters of the alphabet in different fonts. By using this method, a computer and a laser printer can have many different fonts in many different sizes available without using a large amount of memory. (Bezier curves were used to produce most of the graphs in this book.)

Here we define Bezier curves and examine some of their properties. At the end of this section some optional material describes the mathematical construction of Bezier curves.

Definition: Bezier Curve

The Bezier curve $B(t)$ defined for the four points P_0, P_1, P_2, and P_3 (Fig. 5) is

$$B(t) = (1-t)^3 \cdot P_0 + 3(1-t)^2 \cdot t \cdot P_1 + 3(1-t) \cdot t^2 \cdot P_2 + t^3 \cdot P_3$$

for $0 \le t \le 1$:

$$x(t) = (1-t)^3 \cdot x_0 + 3(1-t)^2 \cdot t \cdot x_1 + 3(1-t) \cdot t^2 \cdot x_2 + t^3 \cdot x_3 \quad \text{and}$$
$$y(t) = (1-t)^3 \cdot y_0 + 3(1-t)^2 \cdot t \cdot y_1 + 3(1-t) \cdot t^2 \cdot y_2 + t^3 \cdot y_3 \ .$$

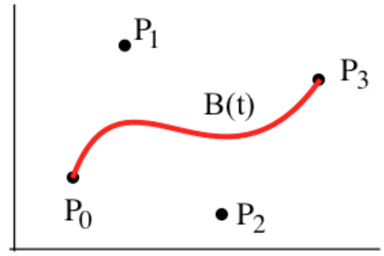

Fig. 5

The four points P_0, P_1, P_2, and P_3 are called **control points** for the Bezier curve. Fig. 6 shows Bezier curves for several sets of control points. The dotted lines connecting the control points in Fig. 6 are shown to help illustrate the relationship between the graph of $B(t)$ and the control points.

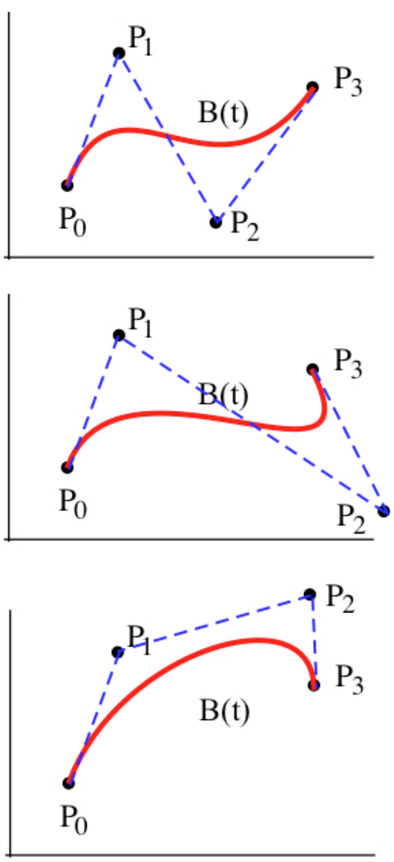

Fig. 6: Several Bezier curves as P_2 is moved

Example 2: Plot the points $P_0 = (0,3)$, $P_1 = (1,5)$, $P_2 = (3,-1)$, and $P_3 = (4,0)$, and determine the equation of the Bezier curve for these control points. Then graph the Bezier curve.

Solution:

$$x(t) = (1-t)^3 \cdot 0 + 3(1-t)^2 \cdot t \cdot 1 + 3(1-t) \cdot t^2 \cdot 3 + t^3 \cdot 4 = -2t^3 + 3t^2 + 3t \ .$$
$$y(t) = (1-t)^3 \cdot 3 + 3(1-t)^2 \cdot t \cdot 5 + 3(1-t) \cdot t^2 \cdot (-1) + t^3 \cdot 0$$
$$= 15t^3 - 24t^2 + 6t + 3 \ .$$

The control points and the graph of $B(t) = (\,x(t), y(t)\,)$ are shown in Fig. 7.

Practice 2: Plot the points $P_0 = (0,4)$, $P_1 = (1,2)$, $P_2 = (4,2)$, and $P_3 = (4,4)$, and determine the equation of the Bezier curve for these control points. Then graph the Bezier curve.

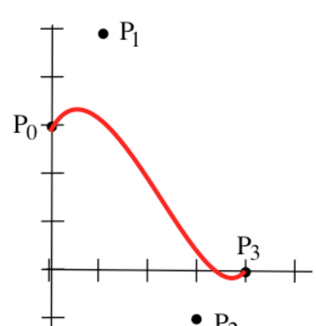

Fig. 7

Properties of Bezier Curves

Bezier curves have a number of properties that make them particularly useful for design work, and some of them are stated below. These properties are verified at the end of this section.

(1) $B(0) = P_0$ and $B(1) = P_3$ so the Bezier curve goes through the points P_0 and P_3 .

This property guarantees that $B(t)$ goes through specified points. If we want two Bezier curves to fit together, it is important that the value at the end of one curve matches the starting value of the next curve. This property guarantees that we can control the values of the Bezier curves at their endpoints by choosing appropriate values for the control points P_0 and P_3 .

(2) B(t) is a cubic polynomial.

This is an important property because it guarantees that B(t) is continuous and differentiable at each point so its graph is connected and smooth at each point. It also guarantees that the graph of B(t) does not "wiggle" too much between control points.

(3) $B'(0)$ = slope of the line segment from P_0 to P_1; $B'(1)$ = slope of the line segment from P_2 to P_3.

This is an important property because it means we can match the ending slope of one curve with the starting slope of the next curve to result in a smooth connection. We can see in Fig. 6 that the dotted line from P_0 to P_1 is tangent to the graph of B(t) at the point P_0.

(4) For $0 \le t \le 1$, the graph of B(t) is in the region whose corners are the control points.

Visually, property (4) means that if we put a rubber band around the four control points P_0, P_1, P_2, and P_3 (Fig. 8), then the graph of B(t) will be inside the rubber banded region. This is an important property of Bezier curves because it guarantees that the graph of B(t) does not get too far from the four control points.

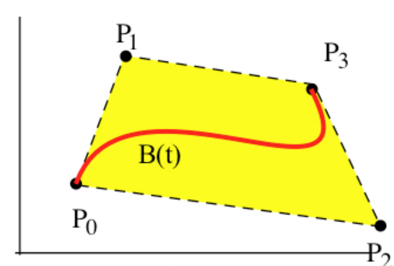

Fig. 8

Example 3: Find a formula for a Bezier curve that goes through the points (1, 1) and (0,0) and is shaped like an "S."

Solution: Since we want the curve to begin at the point (1,1) and end at (0,0) we can put $P_0 = (1,1)$ and $P_3 = (0,0)$. A little experimentation with values for P_1 and P_2 indicates that $P_1 = (-1, 2)$ and $P_2 = (2, -1)$ gives a mediocre "S" shape (Fig. 9). Then the formula for B(t) is

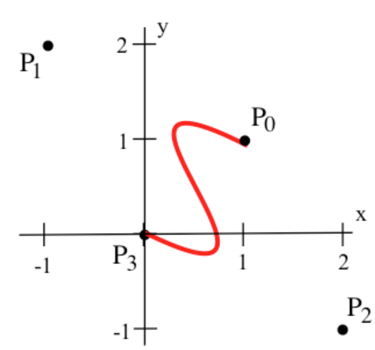

Fig. 9

$$B(t) = (1-t)^3 \cdot P_0 + 3(1-t)^2 \cdot t \cdot P_1 + 3(1-t) \cdot t^2 \cdot P_2 + t^3 \cdot P_3$$

for $0 \le t \le 1$, and

$x(t) = (1-t)^3 \cdot x_0 + 3(1-t)^2 \cdot t \cdot x_1 + 3(1-t) \cdot t^2 \cdot x_2 + t^3 \cdot x_3 = (1-t)^3 \cdot 1 + 3(1-t)^2 \cdot t \cdot (-1) + 3(1-t) \cdot t^2 \cdot 2 + t^3 \cdot 0$

and

$y(t) = (1-t)^3 \cdot y_0 + 3(1-t)^2 \cdot t \cdot y_1 + 3(1-t) \cdot t^2 \cdot y_2 + t^3 \cdot y_3 = (1-t)^3 \cdot 1 + 3(1-t)^2 \cdot t \cdot 2 + 3(1-t) \cdot t^2 \cdot (-1) + t^3 \cdot 0$

Certainly other values of P_1 and P_2 can give similar shapes.

Practice 3: Find a formula for a Bezier curve that goes through the points (0, 0) and (0,1) and is shaped like a "C."

Example 4: Find a formula for a Bezier curve that goes through the point (0, 5) with a slope of 2 and through the point (6, 1) with a slope of 3.

Solution: Since we want to curve to begin at (0,5) and end at (6,1), we put $P_0 = (0,5)$ and $P_3 = (6,1)$. To get the slopes we need to pick P_1 so the slope of the line segment from P_0 to P_1 is 2: going "over 1 and up 2" to get $P_1 = (1,7)$ works fine as do several other points. Similarly, to get the right slope at P_3 we can go "back 1 and down 3" to get $P_2 = (5,-2)$. The Bezier curve for $P_0 = (0,5)$, $P_1 = (1,7)$, $P_2 = (5,-2)$, and $P_3 = (6,1)$ is

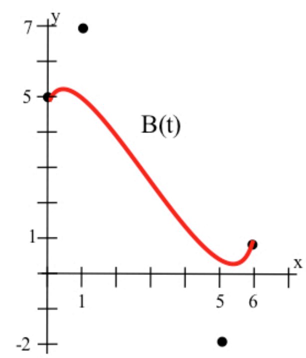

Fig. 10

$x(t) = (1-t)^3 \cdot 0 + 3(1-t)^2 \cdot t \cdot 1 + 3(1-t) \cdot t^2 \cdot 5 + t^3 \cdot 6 = -6t^3 + 9t^2 + 3t$ and

$y(t) = (1-t)^3 \cdot 5 + 3(1-t)^2 \cdot t \cdot 7 + 3(1-t) \cdot t^2 \cdot (-2) + t^3 \cdot 1 = 23t^3 - 33t^2 + 6t + 5.$

The graph of this B(t) is shown in Fig. 10.

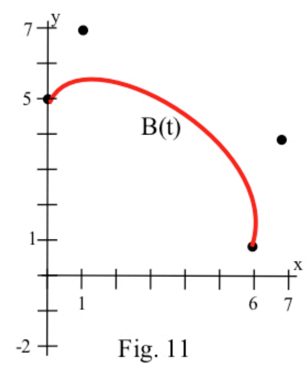

Fig. 11

Keeping the previous values of P_0, P_1 and P_3, we could pick P_2 by going "over 1 and up 3" to $P_2 = (7,4)$, and the graph of the B(t) for this choice of P_2 is shown in Fig. 11. The graph for B(t) when $P_1 = (2,9)$ and $P_2 = (5.5, -0.5)$ is shown in Fig. 12. These, and other choices of P_1 and P_2 satisfy the conditions specified in the problem: the choice of which one you use depends on the other properties of the shape that you want the curve to have.

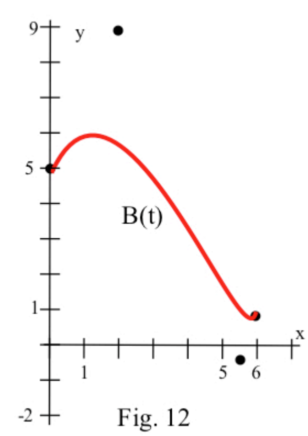

Fig. 12

Practice 4: Find a formula for a Bezier curve that goes through the point (1, 3) with a slope of –2 and through the point (5, 3) with a slope of –1.

Example 5: Find formulas for a pair of Bezier curves so that the first starts at the point A = (0,3) with slope 2, the second ends at the point C = (7,2) with slope 1, and the curves connect at the point B = (4,6) with slope 0.

Solution: For the first Bezier curve B(t) take $P_0 = A = (0,3)$, $P_1 = (1,5)$ (to get the slope 2),

$P_3 = B = (4,6)$, and $P_2 = (3,6)$ (to get the slope 0 at the connecting point). Then,

for $0 \le t \le 1$, $x(t) = (1-t)^3 \cdot 0 + 3(1-t)^2 \cdot t \cdot 1 + 3(1-t) \cdot t^2 \cdot 3 + t^3 \cdot 4$ and

$y(t) = (1-t)^3 \cdot 3 + 3(1-t)^2 \cdot t \cdot 5 + 3(1-t) \cdot t^2 \cdot 6 + t^3 \cdot 6$.

For the second Bezier curve $C(t)$ take $P_0 = B = (4,6)$, $P_1 = (5,6)$ (to get the slope 0 at the connecting point), $P_3 = C = (7,2)$, and $P_2 = (6,1)$ (to get the slope 1). Then,

for $0 \le t \le 1$, $x(t) = (1-t)^3 \cdot 4 + 3(1-t)^2 \cdot t \cdot 5 + 3(1-t) \cdot t^2 \cdot 6 + t^3 \cdot 7$ and

$y(t) = (1-t)^3 \cdot 6 + 3(1-t)^2 \cdot t \cdot 6 + 3(1-t) \cdot t^2 \cdot 1 + t^3 \cdot 2$.

Fig. 13 shows the graphs of $B(t)$ and $C(t)$ and illustrates how they connect, continuously and smoothly, at the common point $(4, 6)$.

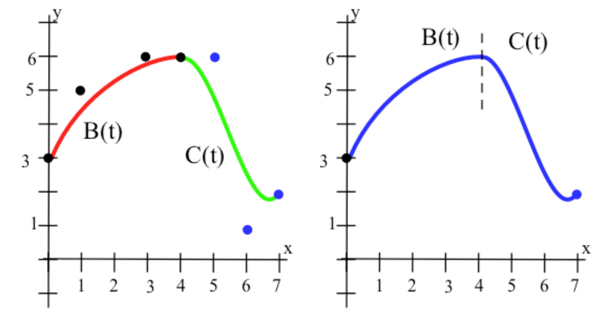

Fig. 13

Using Bezier Curves

In practice, Bezier curves are usually used in computer design or manufacturing programs, and the user of Bezier curves does not have to know the mathematics behind them. But the program creator does!

Typically a designer sketches a crude shape for an object and then moves certain points to locations specified by the plans. Sometimes the designer adds additional points along the curve to "fix" the location of the curve. These "fixed" points along the curve become the endpoints P_0 and P_3 for each of the sections of the curve that will be described by a Bezier formula. Then, for each section of the curve, the designer visually experiments with different locations of the interior control points P_1 and P_2 to get the shape "just right." Meanwhile, the computer program adjusts the formulas for the Bezier curves based on the current locations of the control points for each section, and, when the design is complete, saves the locations of the control points.

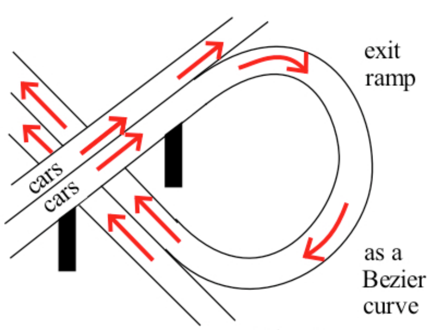

Fig. 14

You may never need to calculate the formulas for Bezier curves (outside of a mathematics class), but if you do any computer–aided design work you will certainly be using these curves. And the ideas and formulas for Bezier curves in two dimensions extend very easily and naturally to describe paths in three dimensions such as the route of a highway exit ramp (Fig. 14) or the path of a hydraulic hose for the landing gear of an airplane.

Mathematical Construction of Bezier Curves & Verifications of Their Properties

In order to use and program Bezier curves we don't need to know where the formulas came from, but their construction is a beautiful piecing–together of simple geometric ideas.

The first idea is the parametric representation of a line segment from point P_A to P_B as $L(t) = (1 - t) \cdot P_A + t \cdot P_B$ for $0 \le t \le 1$.

This parametric pattern for a line is used in the construction of a Bezier curve.

When $t = 0$, the point is $L(0) = P_A$. When $t = 1$, $L(1) = P_B$. When $t = 0.5$, $L(0.5)$ is the midpoint of the line from P_A to P_B (Fig. 15). When $t = 0.2$, $L(0.2)$ is 20% of the way along the line L from P_A to P_B.

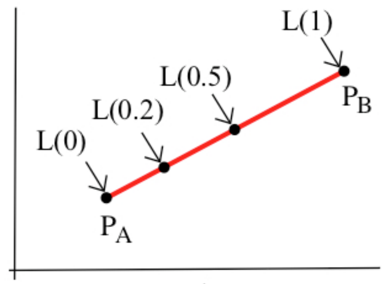

Fig. 15

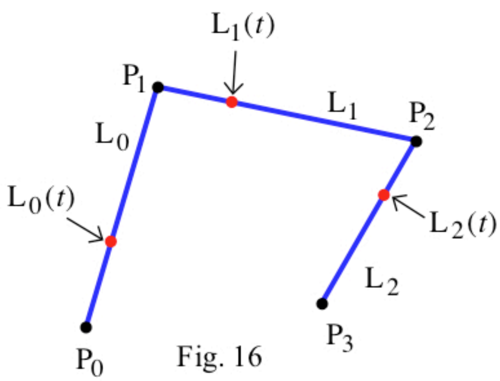

Fig. 16

To construct the Bezier curve for the four control points P_0, P_1, P_2, and P_3 we start by fixing a value of t between 0 and 1. Then we find the point $L_0(t)$ along the parametric line from P_0 to P_1, the point $L_1(t)$ along the parametric line from P_1 to P_2, and the point $L_2(t)$ along the parametric line from P_2 to P_3 (Fig. 16):

$$L_0(t) = (1-t) \cdot P_0 + t \cdot P_1,$$
$$L_1(t) = (1-t) \cdot P_1 + t \cdot P_2,$$
$$L_2(t) = (1-t) \cdot P_2 + t \cdot P_3 .$$

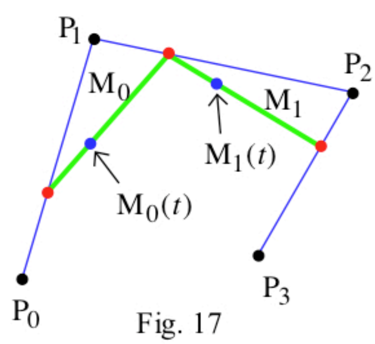

Fig. 17

For the fixed value of t, we then find the point $M_0(t)$ along the parametric line from the point $L_0(t)$ to the point $L_1(t)$, and the point $M_1(t)$ along the parametric line from the point $L_1(t)$ to the point $L_2(t)$ (Fig. 17):

$$M_0(t) = (1 - t) \cdot L_0(t) + t \cdot L_1(t)$$
$$M_1(t) = (1 - t) \cdot L_1(t) + t \cdot L_2(t) .$$

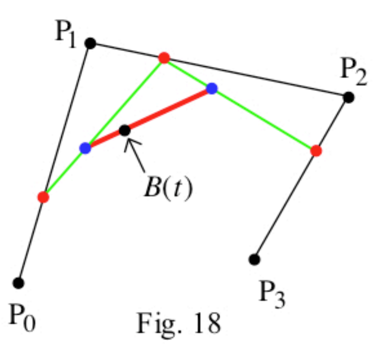

Fig. 18

For the same fixed value of t, we finally find the point B(t) along the parametric line from the point $M_0(t)$ to the point $M_1(t)$ (Fig. 18):

$$B(t) = (1 - t) \cdot M_0(t) + t \cdot M_1(t) .$$

As the variable t takes on different values between 0 and 1, the points $L_0(t)$, $L_1(t)$, and $L_2(t)$ move along the lines connecting P_0, P_1, P_2, and P_3. Similarly, the points $M_0(t)$ and $M_1(t)$ move along the lines connecting $L_0(t)$, $L_1(t)$, and $L_2(t)$, and the point B(t) moves along the line connecting $M_0(t)$ and $M_1(t)$. It is all quite dynamic.

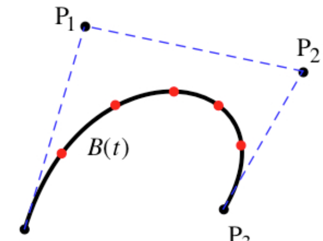

Fig. 19

Fig. 19 shows these points and lines for several values of t between 0 and 1.

We can use the previous geometric construction to obtain the formula for B(t) given in the definition of Bezier curves by working backwards from $B(t) = (1-t)\cdot M_0(t) + t\cdot M_1(t)$:

$$\begin{aligned}
B(t) &= (1-t)\cdot M_0 + t\cdot M_1 \\
&= (1-t)\cdot\{ (1-t)\cdot L_0 + t\cdot L_1\} + t\cdot\{ (1-t)\cdot L_1 + t\cdot L_2\} && \text{replacing } M_0 \text{ and } M_1 \text{ in terms of } L_0, L_1 \text{ and } L_2 \\
&= (1-t)^2\cdot L_0 + 2(1-t)\cdot t\cdot L_1 + t^2\cdot L_2 && \text{simplifying} \\
&= (1-t)^2\cdot\{ (1-t)\cdot P_0 + t\cdot P_1 \} + 2(1-t)\cdot t\cdot\{ (1-t)\cdot P_1 + t\cdot P_2 \} + t^2\cdot\{ (1-t)\cdot P_2 + t\cdot P_3 \} \\
& \text{replacing } L_0, L_1 \text{ and } L_2 \text{ in terms of } P_0, P_1, P_2 \text{ and } P_3 \\
&= (1-t)^3\cdot P_0 + 3(1-t)^2\cdot t\cdot P_1 + 3(1-t)\cdot t^2\cdot P_2 + t^3\cdot P_3 && \text{simplifying}
\end{aligned}$$

Verifications of Properties (1) – (4)

Property (1) is easy to verify by evaluating B(0) and B(1):

$B(0) = (1-0)^3\cdot P_0 + 3(1-0)^2\cdot 0\cdot P_1 + 3(1-0)\cdot t^2\cdot P_2 + 0^3\cdot P_3 = P_0$. Similarly,

$B(1) = (1-1)^3\cdot P_0 + 3(1-1)^2\cdot 1\cdot P_1 + 3(1-1)\cdot 1^2\cdot P_2 + 1^3\cdot P_3 = P_3$.

Property (2) is clear from the defining formula for B(t), or we can expand the powers of $1-t$ and t and collect the similar terms to rewrite B(t) as

$B(t) = (-P_0 + 3P_1 - 3P_2 + P_3)\cdot t^3 + (3P_0 - 6P_1 + 3P_2)\cdot t^2 + (-3P_0 + 3P_1)\cdot t + (P_0)$.

Property (3) can be verified using the rewritten form from Property 2,

$x(t) = (-x_0 + 3x_1 - 3x_2 + x_3)\cdot t^3 + (3x_0 - 6x_1 + 3x_2)\cdot t^2 + (-3x_0 + 3x_1)\cdot t + (x_0)$ and

$y(t) = (-y_0 + 3y_1 - 3y_2 + y_3)\cdot t^3 + (3y_0 - 6y_1 + 3y_2)\cdot t^2 + (-3y_0 + 3y_1)\cdot t + (y_0)$.

Then $\dfrac{d\,x(t)}{d\,t} = 3(-x_0 + 3x_1 - 3x_2 + x_3)\cdot t^2 + 2(3x_0 - 6x_1 + 3x_2)\cdot t + (-3x_0 + 3x_1)$ and

$\dfrac{d\,y(t)}{d\,t} = 3(-y_0 + 3y_1 - 3y_2 + y_3)\cdot t^2 + 2(3y_0 - 6y_1 + 3y_2)\cdot t + (-3y_0 + 3y_1)$.

When $t = 0$, $\dfrac{d\,x(t)}{d\,t} = -3x_0 + 3x_1 = -3(x_1 - x_0)$ and $\dfrac{d\,x(t)}{d\,t} = -3(y_1 - y_0)$ so

$$\frac{d\,B(t)}{d\,t} = \frac{d\,y(t)/dt}{d\,x(t)/dt} = \frac{-3(y_1 - y_0)}{-3(x_1 - x_0)} = \frac{y_1 - y_0}{x_1 - x_0} = \text{slope of the line from } P_0 \text{ to } P_1.$$

The verification that B '(1) equals the slope of the line segment from P_2 to P_3 is similar: evaluate x '(1), y '(1) and $B'(1) = \dfrac{y'(1)}{x'(1)}$.

We will not verify Property (4) here, but it follows from the fact that each point on the Bezier curve is a "weighted average" of the four control points. For $0 \le t \le 1$, each of the coefficients $(1-t)^3$, $3(1-t)^2 \cdot t$, $3(1-t) \cdot t^2$ and t^3 is between (or equal to) 0 and 1, and they always add up to 1 (just expand the powers and add them to check this statement).

Problems

In problems 1 – 6, pairs of points, P_A and P_B, are given. In each problem (a) sketch the line segment L from P_A and P_B, and (b) plot the locations of the points L(0.2), L(0.5), and L(0.9) on the line segment in part (a). Finally, (c) determine the equation of the line segment L(t) from P_A and P_B and graph it.

1. P_A and P_B are given in Fig. 20.
2. P_A and P_B are given in Fig. 21.
3. P_A and P_B are given in Fig. 22.
4. P_A and P_B are given in Fig. 23.

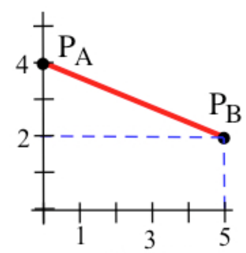

Fig. 20

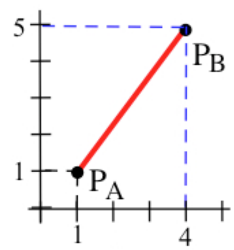

Fig. 21

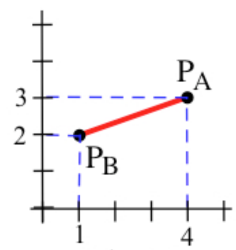

Fig. 22

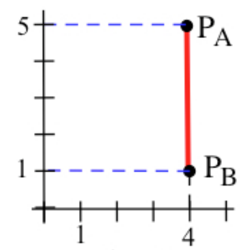

Fig. 23

5. $P_A = (1, 4)$ and $P_B = (5, 1)$.
6. $P_A = (8, 5)$ and $P_B = (4, 3)$.
7. Show that the parametric equations for the line segment given in Example 1,

   $x(t) = (1-t) \cdot x_0 + t \cdot x_1$ and $y(t) = (1-t) \cdot y_0 + t \cdot y_1$ for $0 \le t \le 1$,

   is equivalent to the parametric equations

   $x(t) = x_0 + t \cdot \Delta x$ and $y(t) = y_0 + t \cdot \Delta y$ for $0 \le t \le 1$ where $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$.

In problems 8 – 13, find the parametric equations for a Bezier curve with the given control points or the given properties.

8. $P_0 = (1, 0)$, $P_1 = (2, 3)$, $P_2 = (5, 2)$, $P_3 = (6, 3)$
9. $P_0 = (0, 5)$, $P_1 = (2, 3)$, $P_2 = (1, 4)$, $P_3 = (4, 2)$
10. $P_0 = (5, 1)$, $P_1 = (3, 3)$, $P_2 = (3, 5)$, $P_3 = (2, 1)$
11. $P_0 = (6, 5)$, $P_1 = (6, 3)$, $P_2 = (2, 5)$, $P_3 = (2, 0)$
12. $P_0 = (0, 1)$, $P_3 = (4, 1)$ and $B'(0) = 1$, $B'(1) = -3$
13. $P_0 = (5, 1)$, $P_3 = (1, 3)$ and $B'(0) = 2$, $B'(1) = 3$

In problems 14 – 17, sets of control points P_0, P_1, P_2, and P_3 are shown. Sketch a reasonable Bezier curve for the given control points.

14. P_0, P_1, P_2, and P_3 are given in Fig. 24.

15. P_0, P_1, P_2, and P_3 are given in Fig. 25.

16. P_0, P_1, P_2, and P_3 are given in Fig. 26.

17. P_0, P_1, P_2, and P_3 are given in Fig. 27.

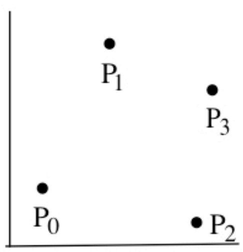

Fig. 24

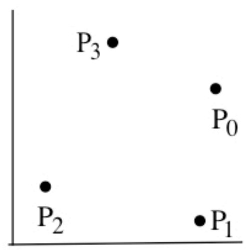

Fig. 25

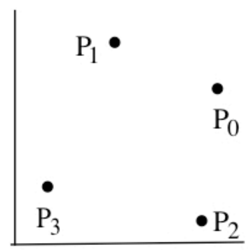

Fig. 26

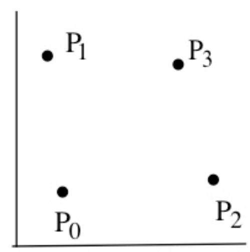

Fig. 27

In problems 18 – 21, sets of control points P_0, P_1, P_2, and P_3 are shown as well as a curve C(t). For each problem explain why we can be certain that C(t) is NOT the Bezier curve for the given control points (state which property or properties of Bezier curves C(t) does not have).

18. P_0, P_1, P_2, and P_3 are given in Fig. 28.

19. P_0, P_1, P_2, and P_3 are given in Fig. 29.

20. P_0, P_1, P_2, and P_3 are given in Fig. 30.

21. P_0, P_1, P_2, and P_3 are given in Fig. 31.

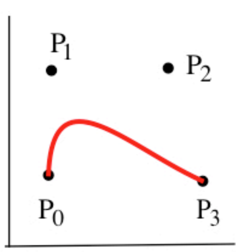

Fig. 28

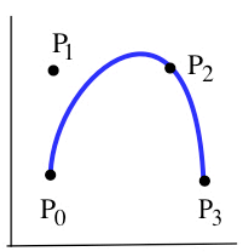

Fig. 29

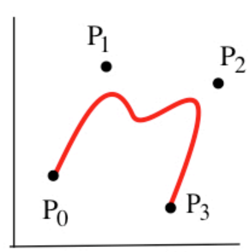

Fig. 30

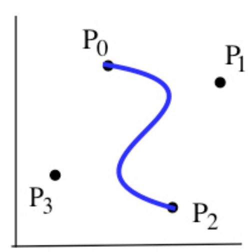

Fig. 31

In problems 22 – 23, find a pair of Bezier curves that satisfy the given conditions.

22. $B(0) = (0, 5)$, $B'(0) = 2$, $B(1) = C(0) = (3, 1)$, $B'(1) = C'(0) = 2$, $C(1) = (6, 2)$, and $C'(1) = 4$.

23. $B(0) = (0, 5)$, $B'(0) = 2$, $B(1) = C(0) = (3, 1)$, $B'(1) = C'(0) = 2$, $C(1) = (6, 2)$, and $C'(1) = 4$

Some Applications of Bezier Curves

The following Applications illustrate just a few of the wide variety of design applications of Bezier curves. This combination of differentiation and algebra is very powerful.

Applications

For each of the following applications, write the equation of a Bezier curve B(t) that satisfies the requirements of the application, and then use your calculator/computer to graph B(t).
(Typically in these applications, the starting and ending points, P_0 and P_3, are specified, but several choices of the control points P_1 and P_2 meet the requirements of the application. Select P_1 and P_2 so the resulting graph of B(t) satisfies the requirements of the application and is also "visually pleasing.")

1. You have been hired to design an escalator for a shopping mall, and the design requirements are that the entrance and exit of the escalator must be horizontal (Fig. 32), the total rise is 20 feet, and the total run is 30 feet.
   (a) Find the equation of a Bezier curve B(t) that meets these design requirements and graph it.
   (Suggestion: Place the origin at the lower left end of the escalator.)

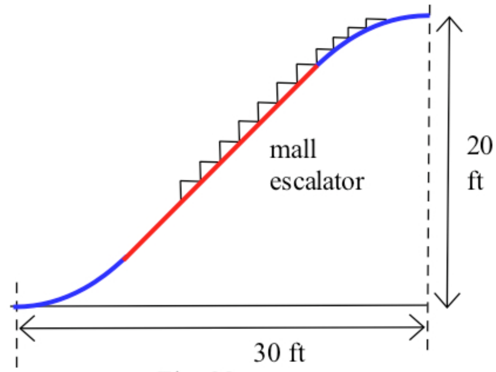

Fig. 32

   (b) In practice, the middle section of an escalator is straight, and each end consists of a curved section (a Bezier curve) that smoothly converts our horizontal motion to motion along the straight section and then to horizontal motion again for our exit. Find the equation of a Bezier curve that models the curve at the entrance to the up escalator (Fig. 33), and the equation of the straight line section.

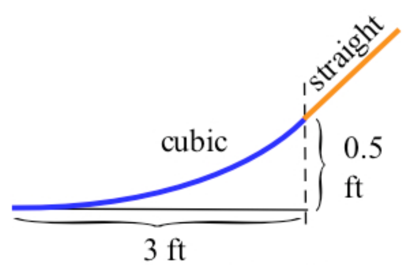

Fig. 33

2. To design the exit ramp on a highway (Fig. 34), you need to find the equation of a Bezier curve so that at the beginning of the exit the elevation is 20 feet with a slope of –0.05 (about 3°), and 600 feet later, measured horizontally, the elevation is 0 feet and the ramp is horizontal.

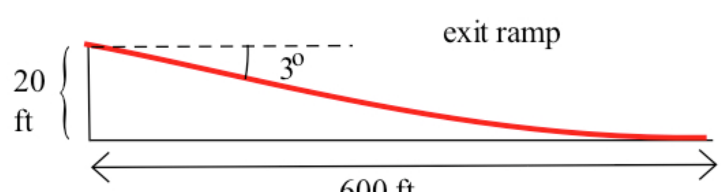

Fig. 34

3. Find a Bezier curve that describes the final 60 feet of the ski jump shown in Fig. 35.

4. Find a Bezier curve that describes the left half of the arch shown in Fig. 36.

5. Find two Bezier curves B(t) and C(t) that describe the pieces of the curve for the top half of the hull of the Concordia Yawl that is 40 feet long, has a beam of 10 feet, and a transom width of 2 feet (Fig. 37).

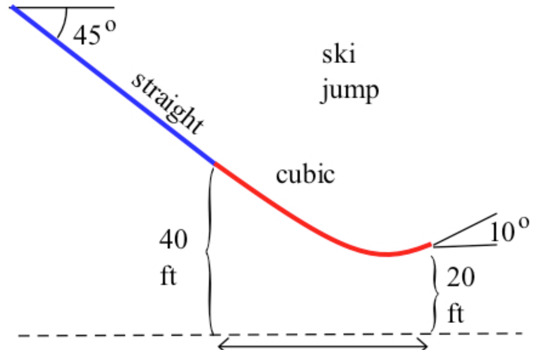

Fig. 35

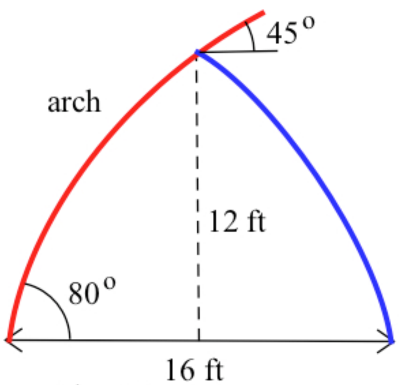

Fig. 36

A

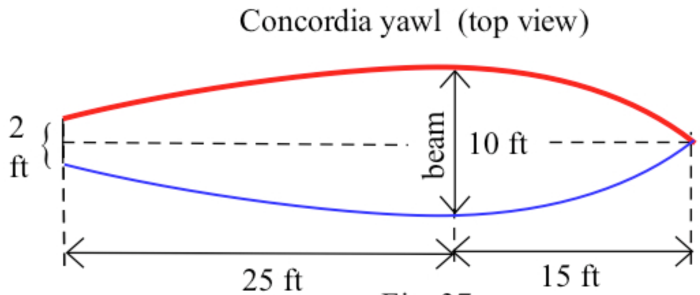

Fig. 37

Final Note: The examples and applications given here have consisted of only one or two Bezier curves, and they are intended only as an introduction to the ideas and techniques of fitting Bezier curves to particular situations. But these ideas extend very nicely to curves that require pieces of several different Bezier curves for a good fit (Fig. 38) and even to curves and surfaces in three dimensions. Unfortunately, the systems of equations tend to grow very large for these extended applications.

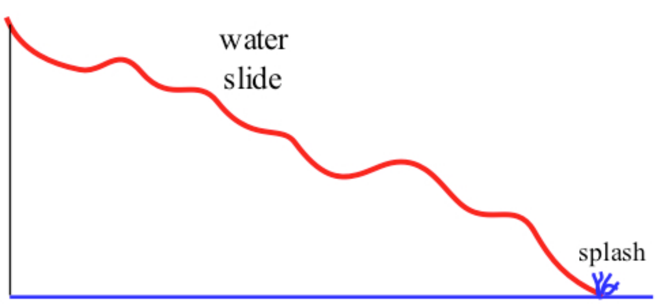

Fig. 38

Practice Answers

Practice 1: (a) $x(t) = (1-t) \cdot x_0 + t \cdot x_1 = (1-t) \cdot 1 + t \cdot 5 = 1 + 4t$,

$y(t) = (1-t) \cdot y_0 + t \cdot y_1 = (1-t) \cdot 2 + t \cdot 4 = 2 + 2t$.

(b) $x(t) = 5 - 4t$, $y(t) = 4 - 2t$

(c) $x(t) = 6 - 3t$, $y(t) = -2 + 3t$

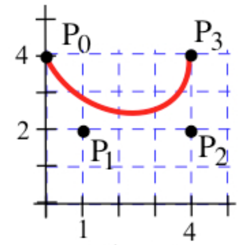

Fig. 39

Practice 2: $P_0 = (0, 4)$, $P_1 = (1, 2)$, $P_2 = (4, 2)$, and $P_3 = (4, 4)$. Then

$x(t) = (1-t)^3 \cdot 0 + 3(1-t)^2 \cdot t \cdot 1 + 3(1-t) \cdot t^2 \cdot 4 + t^3 \cdot 4 = -5t^3 + 6t^2 + 3t$.

$y(t) = (1-t)^3 \cdot 4 + 3(1-t)^2 \cdot t \cdot 2 + 3(1-t) \cdot t^2 \cdot 2 + t^3 \cdot 4 = 6t^2 - 6t + 4$.

The control points and the graph of $B(t) = (x(t), y(t))$ are shown in Fig. 39.

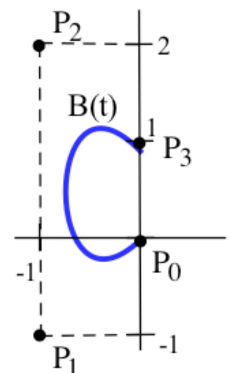

Fig. 40

Practice 3: Take $P_0 = (0, 0)$ and $P_3 = (0, 1)$ to get the correct endpoints.

Take $P_1 = (-1, -1)$ and $P_2 = (-1, 2)$ to get a "C" shape. Then

$x(t) = (1-t)^3 \cdot 0 + 3(1-t)^2 \cdot t \cdot (-1) + 3(1-t) \cdot t^2 \cdot (-1) + t^3 \cdot 0$

$y(t) = (1-t)^3 \cdot 0 + 3(1-t)^2 \cdot t \cdot (-1) + 3(1-t) \cdot t^2 \cdot 2 + t^3 \cdot 1$.

The control points and the graph of $B(t) = (x(t), y(t))$ are shown in Fig. 40.

Practice 4: Take $P_0 = (1, 3)$ and $P_3 = (5, 3)$ to get the correct endpoints.

Take $P_1 = (2, 1)$ and $P_2 = (4, 4)$ to get the correct slopes. Then

$x(t) = (1-t)^3 \cdot 1 + 3(1-t)^2 \cdot t \cdot 2 + 3(1-t) \cdot t^2 \cdot 4 + t^3 \cdot 5$

$y(t) = (1-t)^3 \cdot 3 + 3(1-t)^2 \cdot t \cdot 1 + 3(1-t) \cdot t^2 \cdot 4 + t^3 \cdot 3$.

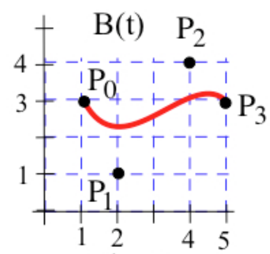

Fig. 41

The control points and the graph of $B(t) = (x(t), y(t))$ are shown in Fig. 41.

Other choices for P_1 and P_2 can also yield correct slopes, and then we have different formulas for $x(t)$ and $y(t)$