13.8 The Chain Rule for Functions of Several Variables

In Section 2.4 we saw the Chain Rule for a function of one variable.

> **Chain Rule (Leibniz notation form)**
>
> If y is a differentiable function of u, and u is a differentiable function of x,
>
> then y is a differentiable function of x and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

One interpretation of this Chain Rule is that if x is a signal that is amplified by a factor of 3 by u (du/dx=3) and the signal u gets amplified by a factor of 2 by y (dy/du=3) then the total amplification of x by the combination of u followed by y is by a factor of 6:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2)(3) = 6 .$$

We can also represent this pattern graphically as in Fig. 1.

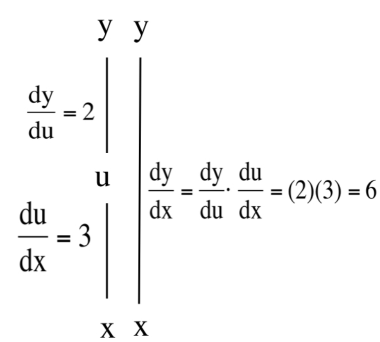


Fig. 1

f is a function of x and y, and each of x and y is a function of t

But suppose that the original signal at t is 1 db (decibel) and that there are two intermediate amplifiers x and y that feed into our final amplifier z as in Fig. 2. If x amplifies t by a factor of 3 (dx/dt=3), z amplifies x by a factor of 2 (dz/dx=2), y amplifies t by a factor of 4 (dy/dt=4), and z amplifies y by a factor of 5 (dz/dy=5), then we can ask what is the total amplification of the original signal 1 db signal at t to the final output z. The original 1 db signal at t becomes 6 db (along the txz path) and 20 db (along the tyz path) for a total output of 26 db.

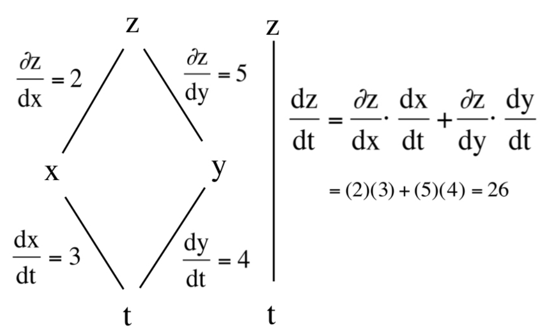
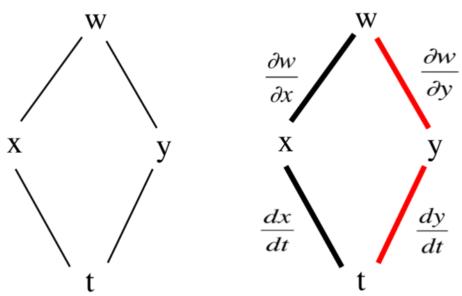


Fig. 2

This is essentially the Chain Rule for a function of two variables: we multiply the rates of change along each path and then add the results to get the total rate of change.

> **The Chain Rule for a Function of Two Dependent Variables**
>
> If $z = f(x, y)$ is a differentiable function of x and y, and x(t) and y(t) are differentiable functions of t, then z is a differentiable function of t, and
>
> $$\frac{df}{dt} = \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} .$$

A tree diagram (Fig. 3) is a visual way to organize information that may help you remember "which derivatives go where" in the Chain Rule formula. Write z, the name of the first function at the top of the diagram. Draw branches to x and y, the names of the independent variables of z. Then draw more branches to t, the independent variable of x and y. Then add in the derivative notations as shown in the diagram below.



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot \frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot \frac{dy}{dt}$$

Fig. 3

The two paths from z to t indicate the Chain Rule formula will have two terms. The number of pieces in a path from z to t indicates the number of factors in the corresponding term. The two paths from z to t and two pieces along each path correspond to the two terms of two factors in the chain rule formula.

Example 1: (a) Use the Chain Rule to find the rate of change of $f(x,y) = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$ when $t = \frac{\pi}{3}$.

(b) If the units of t are seconds, the units of x and y are meters and the units of f are ºC, then what are the units of $\frac{df}{dt}$?

Solution: (a) At $t = \frac{\pi}{3}$ we have $\frac{\partial f}{dx} = y(t) = \sin(\pi/3) = \frac{\sqrt{3}}{2}$, $\frac{\partial f}{dy} = x(t) = \cos(\pi/3) = \frac{1}{2}$,

$\frac{dx}{dt} = -\sin(t) = -\sin(\pi/3) = -\frac{\sqrt{3}}{2}$ and $\frac{dy}{dt} = \cos(t) = \cos(\pi/3) = \frac{1}{2}$ so

$$\frac{df}{dt} = \frac{\partial f}{dx}\cdot \frac{dx}{dt} + \frac{\partial f}{dy}\cdot \frac{dy}{dt} = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}.$$

(b) Clearly the units of $\frac{df}{dt} = \frac{\text{units of f}}{\text{units of t}} = \frac{^oC}{\text{sec}}$. Following the pieces of the Chain Rule we have $\frac{df}{dt} = \frac{\partial f}{dx}\cdot \frac{dx}{dt} + \frac{\partial f}{dy}\cdot \frac{dy}{dt} = \left(\frac{^oC}{m}\right)\left(\frac{m}{s}\right) + \left(\frac{^oC}{m}\right)\left(\frac{m}{s}\right) = \frac{^oC}{s}$ the same result.

Note: In this example we could have simply replaced x and y with the appropriate functions of t so $f(t) = \cos(t)\cdot \sin(t)$ and then differentiated, but such a replacement is not always easy.

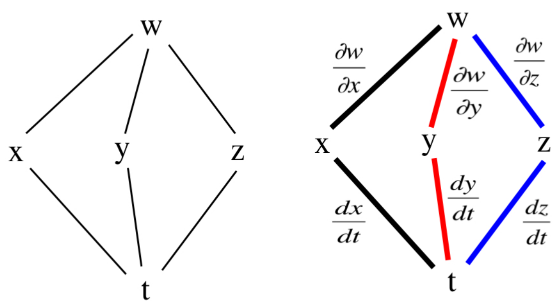
Practice 1: Use the Chain Rule to calculate the value of $\frac{df}{dt}$ when t = 2 for the functions

$$f(x,y) = x^4y^3 + 3x^2y\,,\ x(t) = t^2 + t - 5\,, y(t) = t^3 - 2t^2 - t + 4\,.$$

The Chain Rule for Functions of Three Dependent Variables adds one term to our previous pattern.

> If $w = f(x, y, z)$ is a differentiable function of x, y and z, and x, y, and z are differentiable functions of t, then w is a differentiable function of t, and
>
> $$\frac{df}{dt} = \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Many students find that using a tree diagram (Fig/ 4) for this situation makes it easy to keep track of the pattern: multiply the derivatives along each of the three paths and then add those results together.



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

Fig. 4

<u>Example 2</u>: Use the Chain Rule to find the value of $\frac{df}{dt}$ for $f(x, y, z) = xy + z$ along the helix $x(t) = \cos(t)$, $y(t) = \sin(t)$, $z(t) = t$. What is the derivative's value at $t = 0$? (This is the instantaneous rate of change as our point moves along a helix.)

Solution: If $t = 0$ then $x=1, y=0, z=0$, $\frac{dx}{dt} = -\sin(t) = 0$, $\frac{dy}{dt} = \cos(t) = 1$, and $\frac{dz}{dt} = 1$. At $(1, 0, 0)$

$\frac{\partial f}{\partial x} = y = 0$, $\frac{\partial f}{\partial y} = x = 1$ and $\frac{\partial f}{\partial z} = 1$. Finally,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = (0)(0) + (1)(1) + (1)(1) = 2.$$

Practice 2: Use the Chain Rule to find the value of $\frac{df}{dt}$ for the Example 2 functions when $t = \pi/3$?

In General

There are many ways in which functions of several variables can be combined. Rather than stating or memorizing a Chain Rule pattern for each new situation, just keep in mind the general pattern:

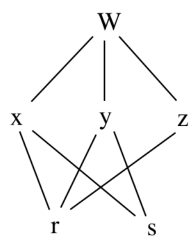
> **Build a tree dependency diagram,**
> **multiply along each path and add these path results.**

Example 3: Suppose $W = f(x,y,z)$, $x = x(r,s)$, $y = y(r,s)$, and $z = z(r)$ and that all of these functions are differentiable,

(a) Build a tree dependency diagram for these functions,

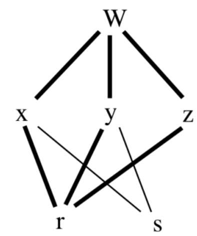
(b) Create a Chain Rule for $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.

Solution:



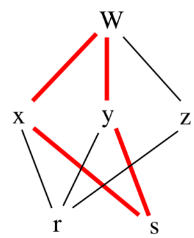
dependency tree

Fig. 5



Three paths from r to W

$$\frac{\partial W}{\partial r} = \frac{\partial W}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial W}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial W}{\partial z}\cdot\frac{\partial z}{\partial r}$$



Two paths from s to W

$$\frac{\partial W}{\partial s} = \frac{\partial W}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial W}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Practice 3: Suppose $T = f(x, y, z)$, $x = x(p, q, r)$, $y = y(p, q)$ and $z = z(p, r)$ and that all of these functions are differentiable.

(a) Build a dependency tree diagram for these functions.

(b) Create Chain Rules for $\frac{\partial T}{\partial p}$ and $\frac{\partial T}{\partial q}$.

Example 4: (a) The voltage V in a circuit satisfies the law V = IR. Write the Chain Rule for $\frac{dV}{dt}$.

(b) If the voltage is dropping because the battery is wearing out and the resistance s increasing because the circuit is heating up, then how fast is the current I changing when R = 500 ohms, I = 0.04 amps, dR/dt = 0.5 ohms/sec, and dV/dt = -0.01 volt/sec?

Solution: (a) $\frac{dV}{dt} = \frac{\partial V}{\partial I}\cdot\frac{\partial I}{\partial t} + \frac{\partial V}{\partial R}\cdot\frac{\partial R}{\partial t}$

(b) $\frac{\partial V}{\partial I} = \frac{\partial (IR)}{\partial I} = R = 500 \ \ ohms$ and $\frac{\partial V}{\partial R} = \frac{\partial (IR)}{\partial R} = I = 0.04 \ \ amps$.

Putting all of this information into the equation in part (a) we have

$$\left(-0.01\frac{amp\cdot ohms}{\sec}\right) = (500 \ \ ohms)\cdot\left(\frac{\partial I}{\partial t}\frac{amps}{\sec}\right) + (0.04 \ \ amps)\cdot\left(0.5\frac{ohms}{\sec}\right) \text{ so}$$

$\frac{\partial I}{\partial t}$ = -0.00006 amps/sec.

PROBLEMS

In problems 1 and 2, use the information in Table 1.

1. Calculate $\frac{df}{dt}$ when t = 1 and t = 3

2. Calculate $\frac{df}{dt}$ when t = 2 and t = 4

$\frac{\partial f}{\partial x} =$

x\y	0	1	2	3
0	5	4	6	2
1	1	9	7	10
2	3	5	4	11
3	7	8	5	13

$\frac{\partial f}{\partial y} =$

x\y	0	1	2	3
0	3	7	2	4
1	6	1	8	3
2	1	4	5	6
3	5	2	9	7

t	1	2	3	4
x	2	0	1	3
y	3	1	0	2

t	1	2	3	4
$\frac{dx}{dt}$	-1	5	-2	6
$\frac{dy}{dt}$	-3	7	-1	8

Table 1

In exercises 37, express $\frac{df}{dt}$ as a function of t by using the Chain Rule. Then evaluate $\frac{df}{dt}$ at the given value of t.

3. $f(x,y) = x^2 + y^2$, $x = \cos t$, $y = \sin t$, $t = \pi$

4. $f(x,y) = x^2y^2 + 3x + 4y + 1$, $x = 3 + t^2$, $y = 1 + 2t$, $t = 2$

5. $f(x,y,z) = x^2y + yz + xz$, $x = 3 + 2t$, $y = t^2$, $z = 5t$, $t = 2$

6. $f(x,y,z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, $x = 1 + 2t$, $y = 2 + 3t$, $z = 3 + 4t$, $t = 1$

7. $f(x,y,z) = 2ye^x - \ln z$, $x = \ln(t^2 + 1)$, $y = \tan^{-1} t$, $z = e^t$, $t = 1$

8. $f(x,y,z) = xyz$, $x = 2\cos(t)$, $y = \sin(t)$, $z = 3t$, $t = \pi$

9. $w = xy + yz + xz$, $x = u + v$, $y = u - v$, $z = uv$, $(u, v) = (-2, 0)$

Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ as functions of u and v by using the Chain Rule.
Then evaluate $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at the point $(u, v) = (-2, 0)$.

10. Find $\frac{\partial w}{\partial r}$ when $r = 1$, $s = -1$, if $w = (x + y + z)^2$, $x = r - s$, $y = \cos(r + s)$, $z = \sin(r + s)$.

11. Find $\frac{\partial z}{\partial u}$ when $u = 0$, $v = 0$, if $z = \cos(xy) + x\cdot \sin(y)$, $x = u + v + 2$, $y = uv$.

12. The lengths a, b, and c of the edges of a rectangular box are changing with time. At the instant in question, a = 1 meter, b = 2 meters, c = 3 meters, $\frac{da}{dt} = \frac{db}{dt} = 1$ m/sec, and $\frac{dc}{dt} = -3$ m/sec. At what rates are the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing or decreasing in length,?

13. In an ideal gas the pressure P (in kilopascals kPa), the volume V (in liters L), and the temperature T (in kelvin K) satisfy the equation PV = 8.31T. How fast is the pressure changing, dP/dt , when the temperature is 310 K and is decreasing at a rate of 0.2 K/s, and the volume is 80 L and is increasing at a rate of 0.1 L/s?

14. Given: w is a function of x, y, and z; x is function of r; y is a function of r and s; z is a function of s and t; and s is a function of t. Make a tree diagram, and then write a chain rule formula for $\frac{\partial w}{\partial t}$.

Practice Answers

Practice 1: $f(x,y) = x^4y^3 + 3x^2y$, $x(t) = t^2 + t - 5$, $y(t) = t^3 - 2t^2 - t + 4$.

If t = 2, then x = 1 , y = 2, $\frac{dx}{dt} = 2t + 1 = 5$ and $\frac{dy}{dt} = 3t^2 - 4t - 1 = 3$. At x=1, y=2

$\frac{\partial f}{\partial x} = 4x^3y^3 + 6xy = 44$ and $\frac{\partial f}{\partial y} = 3x^4y^2 + 3x^2 = 15$ so $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = (44)(5) + (15)(3) = 265$.

Each Chain Rule problem has a lot of pieces so you need to be organized.

Note: Substituting x(t) and y(t) into f gives

$f(t) = (t^2 + t - 5)^4(t^3 - 2t^2 - t + 4)^3 + 3(t^2 + t - 5)^2(t^3 - 2t^2 - t + 4)$ and that would be a messy derivative.

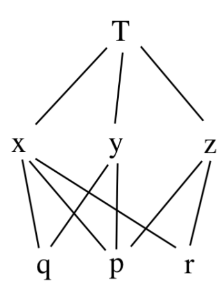
Practice 2: f(x, y, z) = xy + z , x(t) = cos(t), y(t) = sin(t), z(t) = t.

If t = π/3 then x=cos(π/3)= $\frac{1}{2}$, y=sin(π/3)= $\frac{\sqrt{3}}{2}$, z= $\frac{\pi}{3}$, $\frac{dx}{dt} = -\sin(t) = -\sin(\pi/3) = -\frac{\sqrt{3}}{2}$,

$\frac{dy}{dt} = \cos(t) = \cos(\pi/3) = \frac{1}{2}$, and $\frac{dz}{dt} = 1$. At $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$, $\frac{\partial f}{\partial x} = y = \frac{\sqrt{3}}{2}$, $\frac{\partial f}{\partial y} = x = \frac{1}{2}$, and $\frac{\partial f}{\partial z} = 1$.

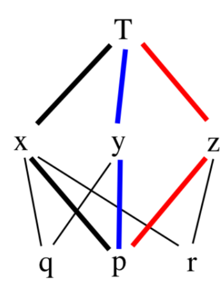
Putting this all together $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + (1)(1) = \frac{1}{2}$.

Practice 3:



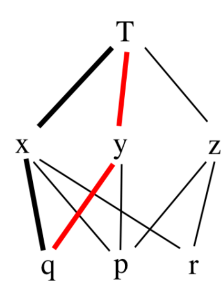
dependency tree

(there is no line from y to r or from z to q)



Three paths from p to T

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial p} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial p} + \frac{\partial T}{\partial z}\cdot\frac{\partial z}{\partial p}$$



Two paths from q to T

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial x}\cdot\frac{\partial x}{\partial q} + \frac{\partial T}{\partial y}\cdot\frac{\partial y}{\partial q}$$

Fig. 6

13.8 Selected Answers

1. (a) When t = 1, x=2 and y=3 so $\frac{\partial f}{\partial x}=11$ and $\frac{\partial f}{\partial y}=6$. Also $\frac{dx}{dt}=-1$ and $\frac{dy}{dt}=-3$.

Then $\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}=(11)\cdot(-1)+(6)\cdot(-3)=-29$.

(b) When t = 3, x=1 and y=0 so $\frac{\partial f}{\partial x}=1$ and $\frac{\partial f}{\partial y}=6$. Also $\frac{dx}{dt}=-2$ and $\frac{dy}{dt}=-1$.

Then $\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}=(1)\cdot(-2)+(6)\cdot(-1)=-8$.

3. When t = π, then x = cos(π)= -1, y = sin(π)= 0, $\frac{dx}{dt}=-\sin(t)=-\sin(\pi)=0$,

$\frac{dy}{dt}=\cos(t)=\cos(\pi)=-1$, $\frac{\partial f}{\partial x}=2x=0$, and $\frac{\partial f}{\partial y}=2y=-2$ so

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}=(0)\cdot(0)+(-2)\cdot(-2)=2$$

5. When t = 2, x = 7, y = 4, z = 10, $\frac{dx}{dt}=2$, $\frac{dy}{dt}=t^2=4$, $\frac{dz}{dt}=5$,

$\frac{\partial f}{\partial x}=2xy+z=66$, $\frac{\partial f}{\partial y}=x^2+z=59$ and $\frac{\partial f}{\partial z}=y+x=11$.

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$=(66)\cdot(2)+(59)\cdot(4)+(11)\cdot(5)=423$$

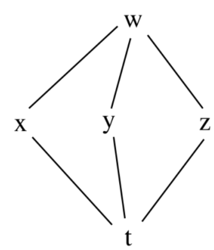


Fig. Problems 5 and 7

7. When t = 1, then $x=\ln(2)$, $y=\tan^{-1}(1)=\frac{\pi}{4}$, $z=e$, $\frac{dx}{dt}=\frac{2t}{1+t^2}=1$, $\frac{dy}{dt}=\frac{1}{1+t^2}=\frac{1}{2}$,

$\frac{dz}{dt}=-\frac{1}{z}=-\frac{1}{e}$, $\frac{\partial f}{\partial x}=2ye^x=2\cdot\frac{\pi}{4}\cdot e^{\ln(2)}=\pi$, $\frac{\partial f}{\partial y}=2e^x=2e^{\ln(2)}=4$ and $\frac{\partial f}{\partial z}=-\frac{1}{e}$.

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dt}=(\pi)\cdot(1)+(4)\cdot\left(\frac{1}{2}\right)+\left(-\frac{1}{e}\right)\cdot(e)=1+\pi$$

9. $w = xy + yz + xz$. When (u, v) = (–2, 0) then x = -2, y = -2, z = 0,

$\frac{dx}{du}=1$, $\frac{dx}{dv}=1$, $\frac{dy}{du}=1$, $\frac{dy}{dv}=-1$, $\frac{dz}{du}=v=0$, $\frac{dz}{dv}=u=-2$

$\frac{\partial w}{\partial x}=y+z=-2$, $\frac{\partial w}{\partial y}=x+z=-2$, $\frac{\partial w}{\partial z}=y+x=-4$

$$\frac{dw}{du}=\frac{\partial w}{\partial x}\cdot\frac{dx}{du}+\frac{\partial w}{\partial y}\cdot\frac{dy}{du}+\frac{\partial w}{\partial z}\cdot\frac{dz}{du}=(-2)(1)+(-2)(1)+(-4)(0)=-4$$

$$\frac{dw}{dv}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dv}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dv}+\frac{\partial w}{\partial z}\cdot\frac{dz}{dv}=(-2)(1)+(-2)(-1)+(-4)(-2)=8$$

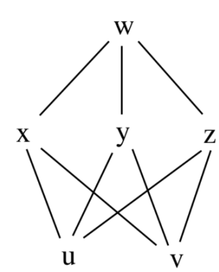


Fig. Problem 9

11. $z = \cos(xy) + x \cdot \sin(y)$, $x = u + v + 2$, $y = u\,v$.

When $u = 0$ and $v = 0$ then $x = 2$, $y = 0$ and $z = 1$.

$$\frac{dx}{du} = 1,\ \frac{dx}{dv} = 1,\ \frac{dy}{du} = v = 0,\ \frac{dy}{dv} = u = 0$$

$$\frac{\partial z}{\partial x} = -y \cdot \sin(xy) + \sin(y) = 0,\ \frac{\partial z}{\partial y} = -x \cdot \sin(xy) + x \cdot \cos(y) = 2$$

$$\frac{dz}{du} = \frac{\partial z}{\partial x} \cdot \frac{dx}{du} + \frac{\partial z}{\partial y} \cdot \frac{dy}{du} = (0)(1) + (2)(0) = 0$$

$$\frac{dz}{dv} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dv} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dv} = (0)(1) + (2)(0) = 0$$

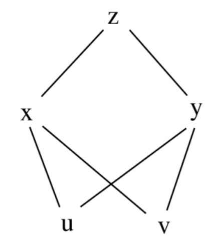


Fig. Problem 11

(That was a lot of work just to get a couple 0s.)

13. We know that $T = 310$ K, $\frac{dT}{dt} = -0.2\ \frac{K}{s}$, $V = 80$ L and $\frac{dV}{dt} = 0.1\ \frac{L}{s}$.

$P = 8.31\frac{T}{V}$ so $\frac{\partial P}{\partial T} = \frac{8.31}{V}$ and $\frac{\partial P}{\partial V} = -8.31\frac{T}{V^2}$. By the Chain Rule

$$\frac{dP}{dt} = \frac{\partial P}{\partial T} \cdot \frac{\partial T}{\partial t} + \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial t} = \left(\frac{8.31}{V}\right) \cdot \left(\frac{\partial T}{\partial t}\right) + \left(-\frac{8.31T}{V^2}\right) \cdot \left(\frac{\partial V}{\partial t}\right)$$

$$= \left(\frac{8.31}{80}\right) \cdot (-0.2) + \left(-\frac{8.31 \cdot 310}{80^2}\right) \cdot (0.1) = -0.061\ \frac{kPa}{\sec}$$