13.6 MAXIMUMS AND MINIMUMS

One important use of derivatives in beginning calculus was to find maximums and minimums of functions of a single variable. Similarly, an important use of partial derivatives is to find maximums and minimums of functions of two (or more) variables.

We are going to consider three situations, and each situation will require a different method.

Three max/min situations: (a) domain = entire xy-plane,
(b) domain = bounded region of the xy-plane, and
(c) domain = a path in the xy-plane.

Situation (a) might ask for the highest elevation anywhere on earth (= at the summit of Mt. Everest = 29,029 ft), (b) might ask for the highest elevation in the state of Washington (= at the summit of Mt. Rainier = 14,4011 ft), and (c) might ask for the highest elevation we achieved during a hike on Mt. Rainier even if we did not reach the summit.

We consider situations (a) and (b) in this section and situation (c) in the next section.

(a) Domain of our max/min search of f is the ENTIRE xy-plane

Definition: A function of two variables f(x,y) has a **local maximum** at (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) in some disk with center (a,b). The value f(a,b) is called a **local maximum value** of f.

The next theorem tells us where we should look for maximums and minimums.

Theorem: If f is differentiable and has a local maximum or minimum at (a,b),
then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Note: It is **possible** that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ and that f(a,b) is **not** a local maximum or minimum. (See Example 2 below.)

Proof: The proof is simply a process of elimination. If one (or both) of $f_x(a,b)$ or $f_y(a,b)$ is positive, then moving a small distance Δ in the direction of that variable will increase the value of f so either $f(a+\Delta,b) > f(a,b)$ or $f(a,b+\Delta) > f(a,b)$ and f(a,b) is not a local maximum. A similar argument also shows that f(a,b) can not be a local minimum.

Example 1: Find all local maximums and minimums of $f(x,y) = x^2 + y^2 - 2x - 6y + 7$.

Solution: $f_x(x,y) = 2x - 2$ so $f_x(x,y) = 0$ when $x = 1$. $f_y(x,y) = 2y - 6$ so $f_y(x,y) = 0$ when $y = 3$.

The only possible location of a maximum or minimum of $f(x,y)$ is at the point $(1,3)$, but we do not know if we have a maximum or minimum or neither at that point. (The graph of $z = f(x,y)$ in Fig. 1 indicates that $f(1,3)$ is a maximum.)

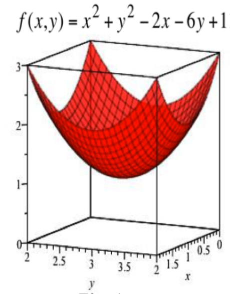

Fig. 1

Example 2: Find all local maximums and minimums of $f(x,y) = x^2 - y^2$.

Solution: $f_x(x,y) = 2x$ so $f_x(x,y) = 0$ when $x = 0$. $f_y(x,y) = -2y$ so $f_y(x,y) = 0$ when $y = 0$.

The only possible location of a maximum or minimum of $f(x,y)$ is at the point $(0,0)$, but $f(0,0) = 0$ is neither a local maximum nor a minimum of f: for any $a \neq 0$, $f(a,0) = a^2 > f(0,0)$ so $f(0,0)$ is not a maximum; for any $b \neq 0$, $f(0,b) = -b^2 < f(0,0)$ so $f(0,0)$ is not a minimum. (The surface $z = x^2 - y^2$ in Fig. 2 is called a "saddle.")

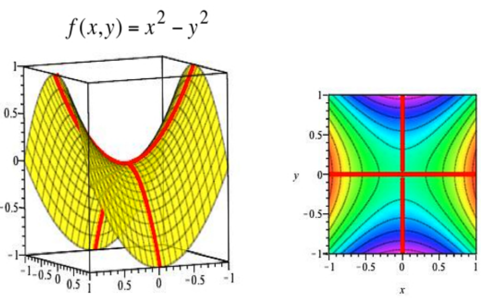

Fig. 2

Example 3: Find all critical points of $f(x,y) = x^2y - 2x^2 - y^3 + 3y + 7$.

Solution: $f_x(x,y) = 2xy - 4x$ and $f_y(x,y) = x^2 - 3y^2 + 3$ so we need to solve the system $\{2xy - 4x = 0$ and $x^2 - 3y^2 + 3 = 0\}$. In order for $0 = 2xy - 4x = 2x(y-2)$ we know that either x=0 or y=2.

x=0 case: Putting x=0 into f_y we have $f_y(0,y) = -3y^2 + 3 = 0$ so $y^2 = 1$ and $y = \pm 1$. This gives us two critical points: (0, 1) and (0, -1).

y=2 case: Putting y=02 into f_y we have $f_y(x,2) = x^2 - 12 + 3 = 0$ so $x^2 = 9$ and $x = \pm 3$. This gives us two new critical points: (3, 2) and (-3, 2).

This function has 4 critical points, and any local maximums or minimums can only occur at one of those 4 locations. Unfortunately it is not easy to decide whether each critical point gives us a local maximum, a local minimum or a saddle. For that we need a graph or the Second Derivative Test.

Practice 1: Find all critical points of $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 + 3$.

In beginning calculus we had a Second Derivative Test to help determine whether a critical point was a local maximum or a local minimum. There is also a Second (Mixed Partial) Derivative Test to help us determine whether a critical point of a function of two variables is a local maximum or minimum or saddle.

Second Derivative Test for Maximums and Minimums

Suppose the second partial derivatives f_{xx}, f_{xy}, f_{yx}, and f_{yy} are continuous in a disk with center (a,b) and $f_x(a,b) = 0$ and $f_y(a,b) = 0$. Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \{ f_{xy}(a,b) \}^2$.

(i) If $D > 0$ and $f_{xx}(a,b) > 0$, then f(a,b) is a local minimum.

(ii) If $D > 0$ and $f_{xx}(a,b) < 0$, then f(a,b) is a local maximum.

(iii) If $D < 0$, then f(a,b) is not a local minimum or a local maximum. (It is a saddle point.)

(iv) If $D = 0$, then the test is "indeterminate": f(a,b) could be a local maximum or a local minimum or neither.

The proof of this theorem is given in an appendix after the Practice solutions.

Example 4: In Example 3 we found 4 critical points of $f(x,y) = x^2y - 2x^2 - y^3 + 3y + 7$: (0, 1), (0, -1), (3, 2) and (-3, 2). Use the Second Derivative Test to determine whether each of these gives a local maximum of f, a local minimum of f, or is a saddle point.

Solution: $f_{xx}(x,y) = 2y - 4$, $f_{yy}(x,y) = -6y$, and $f_{xy}(x,y) = 2x$. Many people find it easiest (and safest) to arrange the numerical information in a table. Fig. 3 shows the surface and level curves for this z = f(x,y).

point	(0, 1)	(0, -1)	(3, 2)	(–3,2)
f_{xx}	-2	-6	0	0
f_{yy}	-6	6	-12	-12
f_{xy}	0	0	6	-6
D	12	-36	-36	-36
result	max	saddle	saddle	saddle

Practice 2: Use the Second Derivative Test to determine whether each critical point of $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 + 3$ (from Practice 1) gives a local maximum of f, a local minimum of f, or is a saddle point.

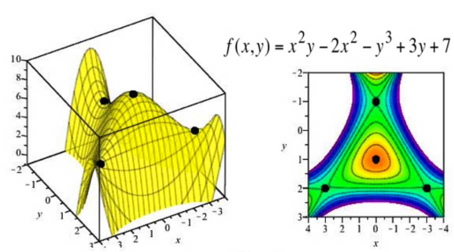

Fig. 3

Example 5: Use the ideas of this section to find the shortest distance from the point (1, 0 –2) to the plane x + 2y + z = 4. (Suggestion: Minimize f(x,y) = the square of the distance)

Solution: The square of the distance of (x,y,z) to the point (1,0,–2) is $(x-1)^2 + (y-0)^2 + (z+2)^2$, is a function of three variables. However, we know that (x,y,z) is on the plane $x + 2y + z = 4$ so $z = 4 - x - 2y$. Replacing z with $4 - x - 2y$ in the distance (squared) formula, we want to minimize $f(x,y) = (x-1)^2 + (y-0)^2 + (6 - x - 2y)^2$.

$f_x = 2(x - 1) + 2(6 - x - 2y)(-1) = 4x + 4y - 14$ and $f_y = 2y + 2(6 - x - 2y)(-2) = 4x + 10y - 24$. We need to find the values of x and y that make f_x and f_y both equal to zero, and the only place that occurs is at the point $(a,b) = (11/6, 5/3)$.

At the point $(a,b) = (11/6, 5/3)$ we have $f_{xx}(a,b) = 4$, $f_{xy} = 4$, and $f_{yy} = 10$ so $D(a,b) = f_{xx}f_{yy} - (f_{xy})^2 = 24 > 0$ and $f_{xx} > 0$. Then by part (i) of the Second Derivative Test we can conclude that f(x,y) has a local minimum at (11/6, 5/3): the shortest distance is

$$\sqrt{f(11/6, 5/3)} = \sqrt{(5/6)^2 + (5/3)^2 + (5/6)^2} = \sqrt{\frac{5}{6}} = \frac{5\sqrt{6}}{6}.$$

(Note: This problem probably would be easier using the ideas from section 11.6.)

(b) Domain of our max/min of f search is a BOUNDED REGION of the xy-plane

In beginning calculus we sometimes needed to find the maximum or minimum value of a function f(x) for x in an interval [a, b]. In that situation we found critical points in [a, b] where $f'(x) = 0$ or was undefined and then we also needed to check the values of f at the ENDPOINTS when x=a and x=b. When looking for max/mins on a bounded region R we still need to find critical points (x,y) in R, but we also need to consider values f on the BOUNDARY of R.

Method for finding Maximums and Minimums on Bounded Domains:

To find the maximum and minimum values of a differential function on a closed bounded region R:

(1) Find the values of f at the critical points of f in R.

(2) Find the extreme values of f on the boundary of R.

(3) The largest value of f from steps (1) and (2) is the absolute maximum value of f on R; the smallest value of f is the absolute minimum value of f on R.

Note: The Second Derivative Test is not used in this situation.

Example 6: Find the absolute maximum and minimum values of $f(x,y) = x^2 - 2xy + 2y + 3$ on the rectangle $R = \{ (x,y) : 0 \le x \le 3 \text{ and } 0 \le y \le 2 \}$.

Solution: Step (1): Find the critical points of f in R. These occur when $f_x(x,y) = 2x - 2y = 0$ and $f_y(x,y) = -2x + 2 = 0$, so (solving algebraically) we have $x = 1$ and $y = 1$. The only critical point from step (1) is $(x,y) = (1,1)$ and $f(1,1) = 4$.

Step (2): Find the critical points and extreme values of f on the boundary of R.

The boundary of the rectangle R consists of four line segments L1, L2, L3, and L4 where L1 = segment from (0,0) to (3,0), L2 = segment from (3,0) to (3,2), L3 = segment from (3,2) to (0,2), and L4 = segment from (0,2) to (0,0).

On L1, $0 \le x \le 3$ and $y = 0$ so $f(x,y) = f(x,0) = x^2 + 3$ which has minimum value $f(0,0) = 3$ and maximum value $f(3,0) = 12$.

On L2, $x = 3$ and $0 \le y \le 2$ so $f(x,y) = f(3,y) = 9 - 6y + 2y + 3 = 12 - 4y$ which has minimum value $f(3,2) = 4$ and maximum value $f(3,0) = 12$.

On L3, $0 \le x \le 3$ and $y = 2$ so $f(x,y) = f(x,2) = x^2 - 4x + 4 + 3 = x^2 - 4x + 7$. Using the methods of Chapter 3 ($f' = 2x - 4 = 0$ when $x = 2$, $f'' = 2 > 0$) we know $f(2,2) = 3$ is a minimum and (checking endpoints $x = 0$ and $x = 3$) that $f(0,2) = 7$ is a maximum.

On L4, $x = 0$ and $0 \le y \le 2$ so $f(0,y) = 2y + 3$ which is a linear function with minimum $f(0,0) = 3$ and maximum $f(0,2) = 7$.

On the boundary (L1, L2, L3, and L4) the minimum value is $3 = f(0,0) = f(2,2)$ and the maximum value is $12 = f(3,0)$.

Comparing the minimum and maximum values from step (2) with $f(1,1) = 4$ from step (1) we have that the absolute minimum is $3 = f(0,0) = f(2,2)$ and the absolute maximum value is $12 = f(3,0)$. See Fig. 4.

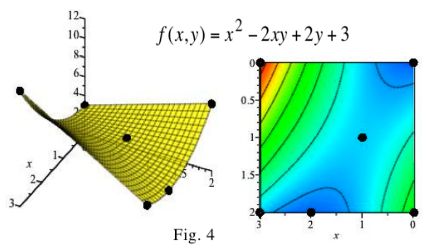

Fig. 4

To find maximums and minimums on a bounded region, we do not use the Second Derivative test, we simply evaluate the function at each critical point and select the largest and smallest values of the function

Practice 3: Find the locations and maximum and minimum values of $f(x,y) = 16 - xy$ on the elliptical domain $2x^2 + y^2 \le 36$.

Example 7: Rewrite the function $f(x,y) = x^2 - 2xy + 2y + 3$ on the boundary lines of the set $D = \{ (x,y) : 0 \le x \le 3 \text{ and } 0 \le y \le 2x \}$

Solution: The boundary of D consists of the three line segments L1 = segment from (0,0) to (3,0), L2 = segment from (3,0) to (3,6), and L3 = the segment from (0,0) to (3,6) along the line $y = 2x$.

On L1, $0 \le x \le 3$ and $y = 0$ so $f(x,y) = f(x,0) = x^2 + 3$.

On L2, $x = 3$ and $0 \le y \le 6$ so $f(x,y) = f(3,y) = 9 - 6y + 2y + 3 = 12 - 4y$.

On L3, $0 \le x \le 3$ and $y = 2x$ so $f(x,y) = f(x,2x) = x^2 - 4x^2 + 4x + 3 = -3x^2 + 4x + 3$.

To actually maximize or minimize f on D, we would now need to apply steps (1) and (2).

PROBLEMS

In problems 1 – 20, find the local maximums, minimums, and saddle points of the given function.

1. $f(x, y) = x^2 + y^2 + 4x - 6y$
2. $f(x, y) = 4x^2 + y^2 - 4x + 2y$
3. $f(x, y) = 2x^2 + y^2 + 2xy + 2x + 2y$
4. $f(x, y) = 1 + 2xy - x^2 - y^2$
5. $g(x,y) = xy^2 - x^3 + 3x - 2y^2 + 5$
6. $g(x,y) = x^3 + 2y^2 - xy^2 - 3x + 7$
7. $f(x, y) = x^2 + y^2 + x^2y + 4$
8. $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$
9. $f(x, y) = x^3 - 3xy + y^3$
10. $f(x, y) = y\sqrt{x} - y^2 - x + 6y$
11. $f(x, y) = xy - 2x - y$
12. $f(x, y) = xy(1 - x - y)$
13. $f(x, y) = \dfrac{x^2y^2 - 8x + y}{xy}$
14. $f(x, y) = x^2 + y^2 + \dfrac{1}{x^2y^2}$
15. $f(x, y) = e^x \cdot \cos(y)$
16. $f(x, y) = (2x - x^2)(2y - y^2)$
17. $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$
18. $f(x, y) = xy \cdot e^{(-x^2 - y^2)}$
19. $f(x, y) = x \cdot \sin(y)$
20. $f(x,y) = 2x^3 + y^2 - 6xy + 10$

In problems 21 – 32 find the maximum and minimum values of f on the set R.

21. $f(x,y) = 5 - 3x + 4y$, R is the closed triangular region with vertices (0,0), (4,0), and (4,5).

22. $f(x,y) = x^2 + 2xy + 3y^2$, R is the closed triangular region with vertices (–1,1), (2,1), and (–1,–2).

23. $f(x,y) = x^2 + y^2 + x^2y + 4$, $R = \{ (x,y) \mid |x| \le 1, |y| \le 1 \}$.

24. $f(x,y) = y\sqrt{x} - y^2 - x + 6y$, $R = (x,y) \mid 0 \le x \le 9, 0 \le y \le 5 \}$.

25. $f(x,y) = 1 + xy - x - y$, R is the region bounded by the parabola $y = x^2$ and the line $y = 4$.

26. $f(x,y) = y^2 - 2x^2 + 10$. R is the region bounded by the parabola $y = x^2$ and the line $y = 4$.

27. $f(x,y) = 2x^2 - y^2 + 30$. R is the region bounded by the parabola $y = x^2$ and the line $y = 4$.

28. $f(x,y) = x^2 + 3y + 7$. R is he region shown in Fig. 5 (include the boundary).

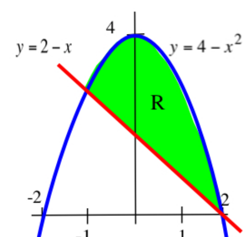

Fig. 5

29. $f(x,y) = x^2 + 3y + 7$. R is the region shown in Fig. 6 (include the boundary).

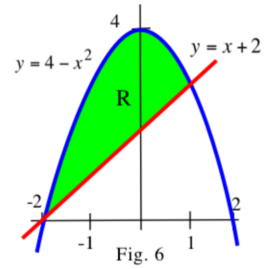

Fig. 6

30. $f(x,y) = 2x^2 + x + y^2 - 2$, $R = \{ (x,y) \mid x^2 + y^2 \le 4 \}$.

31. $f(x,y) = 2x^3 + y^4$, $R = \{ (x,y) \mid x^2 + y^2 \le 1 \}$.

32. $f(x,y) = x^3 - 3x - y^3 + 12y$, R is the quadrilateral whose vertices are $(-2,3)$, $(2,3)$, $(2,2)$, and $(-2,2)$.

33. Find the point on the plane $x + 2y + 3z = 4$ that is closest to the origin.

34. Find the point on the plane $2x - y + z = 1$ that is closest to the point $(-4, 1, 3)$.

35. Find three consecutive numbers whose sum is 100 and whose product is a maximum.

36. Find three positive numbers x, y, and z whose sum is 100 such that $x^a y^b z^c$ is a maximum.

37. Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $9x^2 + 36y^2 + 4z^2 = 36$.

38. Solve the problem in problem 37 for a general ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

39. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $x + 2y + 3z = 6$.

40. Solve the problem in problem 33 for a general plane $x/a + y/b + z/c = 1$ where a, b, and c are positive numbers.

41. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant c.

Practice Solutions

Practice 1: $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 + 3$. $f_x(x,y) = 6x^2 + y^2 + 10x$ and $f_y(x,y) = 2xy + 2y$ so we need to solve the system $\{ 6x^2 + y^2 + 10x = 0$ and $2xy + 2y = 0 \}$. The second equation is easier: in order for 0 = 2xy+2y = 2y(x+1) we know that either y=0 or x=-1.

y=0 case: Putting y=0 into f_x we have $f_x(x, 0) = 6x^2 + 10x = 0$ so $2x(3x+5) = 0$ and x=0 or x= -5/3. This gives us two critical points: (0, 0) and (-5/3, 0).

x= -1 case: Putting x=-1 into f_x we have $f_x(-1, y) = 6 + y^2 - 10 = 0$ so $y^2 = 4$ and $y = \pm 2$. This gives us two new critical points: (-1, 2) and (-1, -2).

This function has 4 critical points: (0, 0), (-5/3, 0), (-1, 2) and (-1, -2) .

Practice 2: $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 + 3$, $f_{xx}(x,y) = 12x + 10$, $f_{yy}(x,y) = 2x + 2$, $f_{xy}(x,y) = 2y$. The information for the Second Derivative Test is organized in the table.

f has a local minimum at (0,0), saddle points at (-1,2) and (-1,-2), and a local maximum at (-5/3, 0).

point	(0, 0)	(-1, 2)	(-1,-2)	(-5/3,0)
f_{xx}	10	-2	-2	-10
f_{yy}	2	0	0	-4/3
f_{xy}	0	4	-4	0
D	20	-16	-16	40/3
result	min	saddle	saddle	max

Practice 3: $f(x,y) = 16 - xy$, $f_x = -y$, $f_y = -x$, so the only interior point with $f_x = f_y = 0$ is (0,0) and f(0,0)=16.

Boundary: $2x^2 + y^2 \le 36$ so $y = \pm\sqrt{36 - 2x^2}$ with $-\sqrt{18} \le x \le \sqrt{18}$. Substituting this y into f we have $f(x,y) = 16 - x \cdot \sqrt{36 - 2x^2}$ which is a function of the single variable x. Then

$$f\,'(x) = \frac{2x^2}{\sqrt{36 - 2x^2}} - \sqrt{36 - 2x^2}$$

. Setting f '(x)=0 and solving for x, we get $x = \pm 3$

so our critical points on the boundary are (3, $\sqrt{18}$), (3, $-\sqrt{18}$), (-3, $\sqrt{18}$), (-3, $-\sqrt{18}$), and the endpoints ($\sqrt{18}$, 0) and ($-\sqrt{18}$, 0) , Evaluating f at each of these critical points we see that the maximum value of f is $16 + 9\sqrt{2} \approx 28.73$ at (3, $-\sqrt{18}$) and (-3, $\sqrt{18}$). The minimum value of f is $16 - 9\sqrt{2} \approx 3.27$ at (3, $\sqrt{18}$) and (-3, $-\sqrt{18}$).

$f(x,y) = 16 - xy$ on $2x^2 + y^2 \le 36$

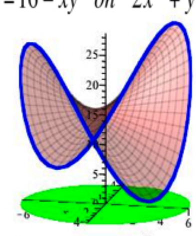

Fig. 7

Appendix: Proof of parts (i) and (ii) of the Second Derivative Test (and part (iv) too)

Suppose we have a critical point (a,b) so $\{ f_x(a,b) = 0 \ \ and \ \ f_y(a,b) = 0 \ \}$. The proof involves calculating the second partial derivative in the direction $u = \langle h,k \rangle$ and then determining when that second partial derivative is positive and negative. The directional derivative in the direction $u = \langle h,k \rangle$ is $D_u f = \nabla f \bullet u = f_x h + f_y k$. Then the second derivative in the direction $u = \langle h,k \rangle$ is

$$\begin{aligned}
D_u^2 f = D_u(D_u f) &= \frac{\partial}{\partial x}(D_u f)h + \frac{\partial}{\partial y}(D_u f)k \\
&= \frac{\partial}{\partial x}(f_x h + f_y k)h + \frac{\partial}{\partial y}(f_x h + f_y k)k \\
&= (f_{xx}h + f_{xy}k)h + (f_{xy}h + f_{yy}k)k \\
&= f_{xx}h^2 + f_{xy}kh + f_{xy}hk + f_{yy}k^2 \\
&= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \\
&= f_{xx}h^2 + 2f_{xy}hk + \frac{f_{xy}^2}{f_{xx}}k^2 + f_{yy}k^2 - \frac{f_{xy}^2}{f_{xx}}k^2 \\
&= f_{xx}\left(h^2 + 2\frac{f_{xy}}{f_{xx}}hk + \frac{f_{xy}^2}{f^2{}_{xx}}k^2 \right) + k^2\left(f_{yy} - \frac{f_{xy}^2}{f_{xx}} \right) \\
&= f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k \right)^2 + \frac{k^2}{f_{xx}}\left(f_{xx}f_{yy} - f_{xy}^2 \right) \\
&= f_{xx}\left\{ \left(h + \frac{f_{xy}}{f_{xx}}k \right)^2 + \frac{k^2}{f_{xx}^2}\left(f_{xx}f_{yy} - f_{xy}^2 \right) \right\}
\end{aligned}$$

If $D = f_{xx}f_{yy} - f_{xy}^2 > 0$ then the part in the curly brackets is positive, so the sign of $D_u^2 f$ is the same as the sign of f_{xx}:

(i) if $f_{xx} > 0$ then $D_u^2 f > 0$ and f is concave up in every direction u so our critical point gives a local minimum

(ii) if $f_{xx} < 0$ then $D_u^2 f < 0$ and f is concave down in every direction u so our critical point gives a local maximum.

Part (iv): $f(x,y) = x^2y^2$, $g(x,y) = -x^2y^2$, $h(x,y) = x^3y^3$ all have the critical point (0,0) and D = 0 at that critical point. But f has a local minimum at (0,0), g has a local maximum at (0,0) and h has neither a local min or max at (0,0). So D = 0 at a critical point does not tell us whether we have a local max or a local min or neither.

13.6 Selected Answers

1. Minimum $f(-2, 3) = -13$

3. Minimum $f(0, -1) = -1$

5. Local maximum: f(1,0)=7. Saddle points at (-1,0), (2,3) and (2,-3)

7. Local minimum: $f(0, 0) = 4$. Saddle points: $(\pm\sqrt{2}, -1)$

9. Local minimum: $f(1, 1) = -1$. Saddle point $f(0, 0) = 0$

11. Saddle point $f(1, 2) = -2$

13. Local maximum $f(-1/2, 4) = -6$

15. None

17. Local maximum $f(0, 0) = 2$, local minimum $f(0, 2) = -2$, saddle points $(\pm 1, 1)$

19. Saddle points $(0, n\pi)$, n and integer

21. Minimum $f(4, 0) = -7$, maximum $f(4, 5) = 13$

23. Maximum $f(-1, 1) = f(1, 1) = 7$, minimum $f(0, 0) = 4$

25. Critical points: f1,1)=0, f(-2,4)=- -9 minimum, f(2,4)= 3 maximum, f(-1/3, 1/9) = 32/27

27. Critical points: f(0,0) = 30, f(1,1)=f(–1,1)=31. maximum, f(–2,4)=f(2,4)=22, f(0.4) = 14 minimum

29. Critical points: f(1,3)=17, f(–2,0)=11, f(0,4) = 19 maximum, f(–3/2, 1/2) = 43/4 minimum

31. Critical points: f(0,0)=0 minimum, $f(0, \pm 1) = 1$, $f(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = \frac{85}{16}$ maximum

33. (2/7, 4/7, 6/7)

35. $\frac{100}{3}$, $\frac{100}{3}$, $\frac{100}{3}$

37. $16/\sqrt{3}$

39. 4/3

41. Cube, edge length c/12