12.2 DERIVATIVES AND ANTIDERIVATIVES OF VECTOR–VALUED FUNCTIONS

Derivatives of Vector–valued Functions

The derivative of a vector–valued function is another vector–valued function, and this derivative is defined much like the derivative of a scalar function. Derivatives of vector–valued functions are generally easy to compute, component–by–component, and they have a useful geometric interpretation as the vectors tangent to the graph of the vector–valued function.

> **Definition:** The **derivative** of $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, denoted $\frac{d}{dt}\mathbf{r}(t)$ or $\mathbf{r}'(t)$, is
>
> $$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = \langle x'(t), y'(t), z'(t) \rangle$$
>
> provided the limit exists and is finite. (Fig. 1)

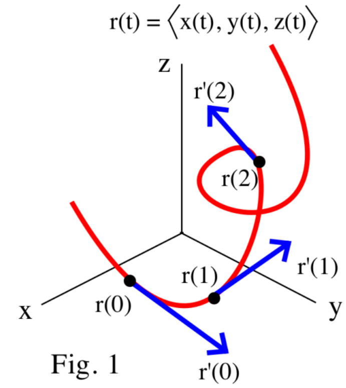

Fig. 1

A vector–valued function $\mathbf{r}(t)$ is differentiable at a point $t = t_0$ if and only if each of its component functions is differentiable at $t = t_0$, and we can calculate the derivative $\mathbf{r}'(t)$ by calculating the three derivatives $x'(t)$, $y'(t)$, and $z'(t)$.

Visualizing $\mathbf{r}'(t)$: If $\mathbf{r}(t)$ is the position of an object at time t, then the difference vector $\mathbf{r}(t+\Delta t) - \mathbf{r}(t)$ represents the **change** in position from time t to time $t + \Delta t$ (Fig. 2), and the ratio $\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$ is a vector measuring the average rate of change of position during the time interval from t to $t + \Delta t$. The limit $\mathbf{r}'(t)$ of this "average rate of change" vector has two useful geometric properties (Fig. 3):

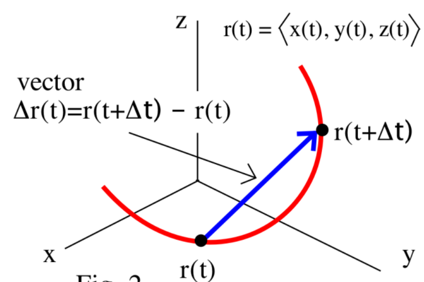

Fig. 2

- $\mathbf{r}'(t)$ is **tangent** to the graph of $\mathbf{r}(t)$, and
- the magnitude of $\mathbf{r}'(t)$ is the **speed** of the object along the path at the time t.

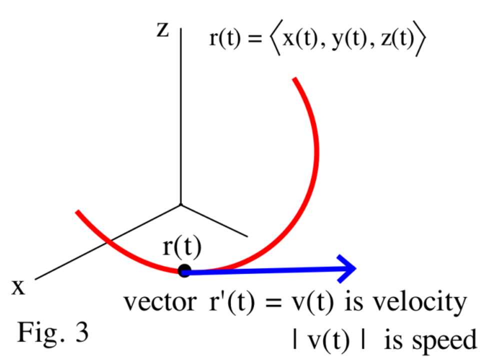

Fig. 3

The vector $\mathbf{r}'(t)$ is called the velocity of $\mathbf{r}(t)$.

The vector $|\mathbf{r}'(t)|$ is called the speed of $\mathbf{r}(t)$.

Definitions: Velocity, Speed, Direction, and Acceleration

If $\mathbf{r}(t)$ is the **position** of an object at time t, then

the **velocity** of the object is $\mathbf{v}(t) = \frac{d}{dt}\mathbf{r}(t) = \mathbf{r}'(t)$ (a vector tangent to $\mathbf{r}(t)$),

the **speed** of the object is $|\mathbf{v}(t)|$ (a scalar) ,

the **direction** of travel is $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$ (the **unit tangent vector**) , and

the **acceleration** is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Example 1: A ladybug is crawling up a helix so its position vector is $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ as shown in Fig. 4.

(a) At the points labeled A, B, and C on the graph of $\mathbf{r}(t)$, estimate the sign (positive or negative) of each component of $\mathbf{r}'(t)$.

(b) The values of t for the point A, B, and C are $t = \pi/6, 3\pi/4$, and $7\pi/4$ respectively. Calculate $\mathbf{r}'(t)$ at the given values of t and compare the results with your estimates in part (a).

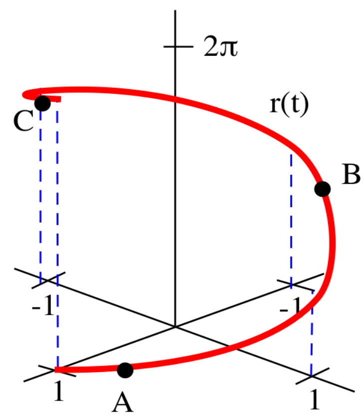

Fig. 4

Solution: (a) At A, $\mathbf{r}'(t)$ is $\langle x'(t), y'(t), z'(t) \rangle = \langle -, +, + \rangle$. At B, $\mathbf{r}'(t)$ is $\langle -, -, + \rangle$.
At C, $\mathbf{r}'(t)$ is $\langle +, +, + \rangle$.

(b) $\mathbf{r}'(\pi/6)$ is $\langle x'(\pi/6), y'(\pi/6), z'(\pi/6) \rangle = \langle -\sin(\pi/6), \cos(\pi/6), 1 \rangle \approx \langle -0.5, 0.867, 1 \rangle$.
$\mathbf{r}'(3\pi/4)$ is $\langle -\sin(3\pi/4), \cos(3\pi/4), 1 \rangle \approx \langle -0.707, -0.707, 1 \rangle$.
$\mathbf{r}'(7\pi/4)$ is $\langle -\sin(7\pi/4), \cos(7\pi/4), 1 \rangle \approx \langle 0.707, 0.707, 1 \rangle$.

Practice 1: The position vector of an object at time t is $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ as shown in Fig. 5. Calculate the position, velocity, speed, direction, and acceleration of the object when t = 0, 1, and 2.

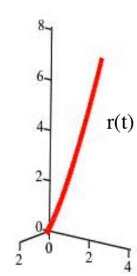

Fig. 5

Angles of Intersection Between Space Curves

The angle of intersection between two curves at a point in space is the angle between their tangent vectors (velocities) at that point of intersection, and the dot product of the tangent vectors can be used to find this angle.

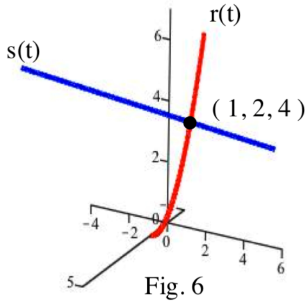

Fig. 6

Example 2: The parabolic path $\mathbf{r}(t) = \langle 1, t, t^2 \rangle$ intersects the line $\mathbf{s}(t) = \langle -2 + 3t, 6 - 4t, 2 + 2t \rangle$ (Fig. 6) at the point (1, 2, 4). Find the angle of intersection of the curves at that point.

Solution: The parabola goes through (1, 2, 4) when t = 2, and the line goes through (1, 2, 4) when t = 1. Then $\mathbf{r}'(2) = \langle 0, 1, 4 \rangle$ and $\mathbf{s}'(1) = \langle 3, -4, 2 \rangle$ so

$$\cos(\theta) = \frac{\mathbf{r}'(2) \cdot \mathbf{s}'(1)}{|\mathbf{r}'(2)|\,|\mathbf{s}'(1)|}$$

$$\approx \frac{4}{\sqrt{17}\sqrt{29}} \approx 0.180 \text{ and}$$

and $\theta \approx 1.390$ (or about 79.6°).

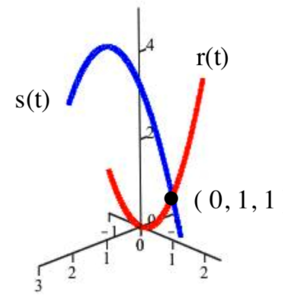

Fig. 7

Practice 2: The parabolic paths $\mathbf{r}(t) = \langle 0, t, t^2 \rangle$ and $\mathbf{s}(t) = \langle 2 - t, 1, 5 - t^2 \rangle$ (Fig. 7) intersect at the point (0, 1, 1). Find the angle of intersection of the curves at that point.

Differentiation of Combinations of Vector–valued Functions

For scalar functions we have patterns for differentiating sums, differences, products, and compositions, and there are similar rules for differentiating combinations of vector–valued functions. In fact, the rules for vector–valued functions are almost identical to the corresponding rules for scalar function derivatives.

Differentiation Patterns for Vector–valued Functions

Constant: If $\mathbf{C}$ is a constant vector, then $\frac{d}{dt}\mathbf{C} = 0$ vector.

If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector–valued functions, k is a scalar number, and f(t) is a scalar function, then

Sum: $\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) = \frac{d}{dt}\mathbf{u}(t) + \frac{d}{dt}\mathbf{v}(t) = \mathbf{u}'(t) + \mathbf{v}'(t)$

Difference: $\frac{d}{dt}(\mathbf{u}(t) - \mathbf{v}(t)) = \frac{d}{dt}\mathbf{u}(t) - \frac{d}{dt}\mathbf{v}(t) = \mathbf{u}'(t) - \mathbf{v}'(t)$

Products: $\frac{d}{dt}(k\mathbf{u}(t)) = k\frac{d}{dt}(\mathbf{u}(t)) = k\,\mathbf{u}'(t)$

scalar $$\frac{d}{dt}(\, f(t)\,\mathbf{u}(t)\,) = f(t)\frac{d}{dt}(\,\mathbf{u}(t)\,) + \frac{d\,f(t)}{dt}\,\mathbf{u}(t) = f(t)\,\mathbf{u}'(t) + f\,'(t)\,\mathbf{u}(t)$$

dot $$\frac{d}{dt}(\,\mathbf{u}(t)\bullet\mathbf{v}(t)\,) = \mathbf{u}(t)\bullet\frac{d\,\mathbf{v}(t)}{dt} + \frac{d\,\mathbf{u}(t)}{dt}\bullet\mathbf{v}(t) = \mathbf{u}(t)\bullet\mathbf{v}'(t) + \mathbf{u}'(t)\bullet\mathbf{v}(t)$$

cross $$\frac{d}{dt}(\,\mathbf{u}(t) \times \mathbf{v}(t)\,) = \mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$$

Chain Rule: $$\frac{d}{dt}\,\mathbf{u}(\, f(t)\,) = f\,'(t)\,\mathbf{u}'(\, f(t)\,)$$

You should notice that all of the **product** differentiation patterns have the form

(first function) "times" (derivative of the second) plus (derivative of the first) "times" (second)

where the "times" is the appropriate type of multiplication, either scalar, dot, or cross.

Proofs: The proofs are very straightforward (componentwise) for the results for the derivatives of constant vectors, sums, differences and a scalar times a vector–valued function, and they are left for you.

The proofs given below for the product rules all follow the pattern of rewriting the original function as components, using our usual product rule or chain rule to differentiate the component functions, and then rewriting the results as the appropriate product of vectors.

Scalar: $$\frac{d}{dt}(\, f(t)\,\mathbf{u}(t)\,) = \left\langle \frac{d}{dt}\, f(t)u_1(t), \frac{d}{dt}\, f(t)u_2(t), \frac{d}{dt}\, f(t)u_3(t) \right\rangle$$

$$= \left\langle f(t)u_1'(t) + f\,'(t)u_1(t),\ f(t)u_2'(t) + f\,'(t)u_2(t),\ f(t)u_3'(t) + f\,'(t)u_3(t) \right\rangle$$

$$= \left\langle f(t)u_1'(t),\ f(t)u_2'(t),\ f(t)u_3'(t) \right\rangle + \left\langle f\,'(t)u_1(t),\ f\,'(t)u_3(t),\ f\,'(t)u_3(t) \right\rangle$$

$$= f(t)\left\langle u_1'(t),\ u_2'(t),\ u_3'(t) \right\rangle + f\,'(t)\left\langle u_1(t),\ u_2(t),\ u_3(t) \right\rangle$$

$$= f(t)\,\mathbf{u}'(t) + f\,'(t)\,\mathbf{u}(t)\,.$$

Dot: $$\frac{d}{dt}(\,\mathbf{u}(t)\bullet\mathbf{v}(t)\,) = \left\langle \frac{d}{dt}\,u_1(t)v_1(t),\ \frac{d}{dt}\,u_2(t)v_2(t),\ \frac{d}{dt}\,u_3(t)v_3(t) \right\rangle$$

$$= \left\langle u_1(t)v_1'(t) + u_1'(t)v_1(t),\ u_2(t)v_2'(t) + u_2'(t)v_2(t),\ u_3(t)v_3'(t) + u_3'(t)v_3(t) \right\rangle$$

$$= \left\langle u_1(t)v_1'(t),\ u_2(t)v_2'(t),\ u_3(t)v_3'(t) \right\rangle + \left\langle u_1'(t)v_1(t),\ u_2'(t)v_2(t),\ u_3'(t)v_3(t) \right\rangle$$

$$= \mathbf{u}(t)\bullet\mathbf{v}'(t) + \mathbf{u}'(t)\bullet\mathbf{v}(t)\,.$$

The pattern for the derivative of a cross product can also be proved by resorting to the definition of the cross product and showing that the components of $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t))$ match the components of $\mathbf{u}(t) \times \mathbf{v}'(t) + \mathbf{u}'(t) \times \mathbf{v}(t)$, but the process is algebraically long and is omitted.

Chain rule: $\frac{d}{dt}\mathbf{u}(f(t)) = \left\langle \frac{d}{dt} u_1(f(t)), \frac{d}{dt} u_2(f(t)), \frac{d}{dt} u_3(f(t)) \right\rangle$

$= \left\langle f'(t)u_1'(f(t)), f'(t)u_2'(f(t)), f'(t)u_3'(f(t)) \right\rangle$

$= f'(t)\left\langle u_1'(f(t)), u_2'(f(t)), u_3'(f(t)) \right\rangle = f'(t)\,\mathbf{u}'(f(t))$.

These differentiation patterns simply provide alternate, and sometimes easier, ways to compute derivatives. Occasionally they are useful for deriving results about the behavior of vector–valued functions such as the one given in the next example.

Example 3: Suppose a differentiable position vector $\mathbf{r}(t)$ of an object has constant length so $|\mathbf{r}(t)| = k$ for all t. Show that the direction of travel of the object is always perpendicular to its position.

Solution: $\mathbf{r}(t)\bullet\mathbf{r}(t) = |\mathbf{r}(t)|^2 = k^2$ for all t, so $\mathbf{r}(t)\bullet\mathbf{r}(t)$ is a constant so $\frac{d}{dt}\mathbf{r}(t)\bullet\mathbf{r}(t) = 0$.

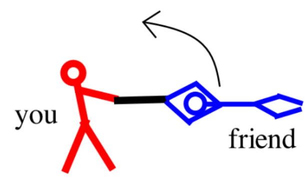

Fig. 8: Twirling a friend

But we also know that

$\frac{d}{dt}\{\mathbf{r}(t)\bullet\mathbf{r}(t)\} = \mathbf{r}(t)\bullet\mathbf{r}'(t) + \mathbf{r}'(t)\bullet\mathbf{r}(t) = 2\mathbf{r}(t)\bullet\mathbf{r}'(t)$,

so we can conclude that $\mathbf{r}(t)\bullet\mathbf{r}'(t) = 0$ for all t.

But $\mathbf{r}(t)\bullet\mathbf{r}'(t) = 0$ means that $\mathbf{r}(t)$ is perpendicular to $\mathbf{r}'(t)$ for all t, and that means the position, $\mathbf{r}(t)$, is always perpendicular to the velocity, $\mathbf{r}'(t)$. The velocity vector $\mathbf{r}'(t)$ points in the direction of travel of the object so we have shown that the direction of travel of the object is always perpendicular to its position.

The result in Example 3 also has straightforward physical interpretations. If we are twirling someone around a central point (Fig. 8), we can take the central point to be the origin. Then the twirled person is always a constant distance from the central point and the magnitude of their position vector is a constant. The result of this example says that the twirled person's velocity vector is always perpendicular to their position vector. If we let go of the person, their motion will be a straight line that is perpendicular to the circular path they were following (Fig. 9). An equivalent situation in three

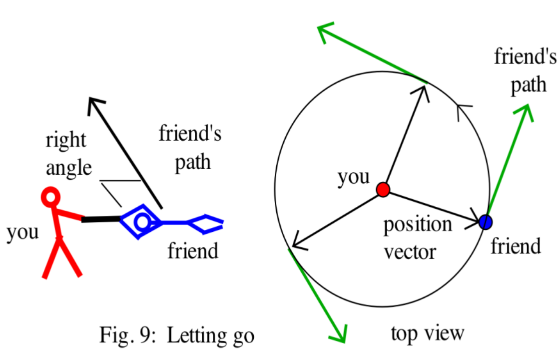

Fig. 9: Letting go

dimensions is an object moving on the surface of a sphere (Fig. 10) such as the earth (almost). If gravity is "turned off," this object will travel along a path perpendicular to the vector from the center of the earth to its position.

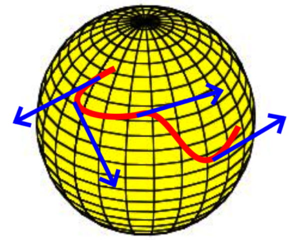
Fig. 10: Path on a sphere

Antiderivatives of Vector–valued Functions

Since the derivative of a vector–valued function is defined to be the vector formed by the derivative of each of the component functions, the antiderivative of a vector–valued function is also defined component by component.

Definition: If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$,

then the antiderivative of $\mathbf{r}(t)$ is $\int \mathbf{r}(t)\,dt = \langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \rangle$

provided that the antiderivatives of $x(t)$, $y(t)$, and $z(t)$ exist.

Example 4: The velocity of an object is $\mathbf{v}(t) = \langle 4t, -\sin(t), e^t \rangle$ and its position at time $t = 0$ is $\mathbf{r}(0) = \langle 2, 3, 4 \rangle$, Find a formula for $\mathbf{r}(t)$, its position at time t.

Solution: $\mathbf{v}(t) = \mathbf{r}'(t)$ so

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\,dt = \langle \int 4t\,dt, \int -\sin(t)\,dt, \int e^t\,dt \rangle = \langle 2t^2 + A, \cos(t) + B, e^t + C \rangle.$$

Then we can use the initial condition that $\mathbf{r}(0) = \langle 2, 3, 4 \rangle$, to determine that $A = 2$, $B = 2$, and $C = 3$ so $\mathbf{r}(t) = \langle 2t^2 + 2, \cos(t) + 2, e^t + 3 \rangle$.

Practice 3: The velocity of an object is $\mathbf{v}(t) = \langle 6t^2, \cos(t), 12e^{3t} \rangle$ and its position at time $t = 0$ is $\mathbf{r}(0) = \langle 1, -5, 2 \rangle$, Find a formula for $\mathbf{r}(t)$, its position at time t.

The inertial guidance system on an airplane uses antiderivatives of vector–valued functions to determine the location of the airplane. The inertial guidance system starts with the initial location and velocity of the airplane and then uses lasers to measure the acceleration of the airplane in each of the x, y, and z directions several times per second. From this acceleration (change in velocity) data, the computer in the system calculates the new velocity in each direction several times per second and then uses the velocities (changes in positions) to calculate the new position of the airplane relative to the starting position.

Example 5: Fig. 11 shows the initial acceleration, velocity and position of an object along the x–axis as well as its acceleration at 1 second time intervals. Fill in the empty spaces in the table and determine the position of the object on the x–axis after 9 seconds.

time (sec)	acceleration (ft/sec)	velocity (ft/sec)	position (ft)
0	0	2	5
1	4		
2	6		
3	4		
4	2		
5	8		
6	6		
7	0		
8	0		
9	0		

Fig. 11

Solution: Acceleration is the $\frac{\text{change in velocity}}{\text{change in time}} = \frac{\text{change in velocity}}{\text{1 second}}$ so each entry in the velocity column is the previous velocity plus the change in velocity (acceleration):

at t = 1, velocity = (previous velocity) + (change in velocity) = 2 + 4 = 6

at t = 2, velocity = (previous velocity) + (change in velocity) = 6 + 6 = 12

at t = 3, velocity = (previous velocity) + (change in velocity) = 12 + 4 = 16.

The rest of the entries in the velocity column are calculated in the same way and the velocity values are shown in Fig. 12.

time (sec)	acceleration (ft/sec)	velocity (ft/sec)	position (ft)
0	0	2	5
1	4	6	11
2	6	12	23
3	4	16	39
4	2	18	57
5	8	26	83
6	6	32	115
7	0	32	147
8	0	32	179
9	0	32	211

Fig. 12

Velocity is

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\text{change in position}}{\text{1 second}}$$

so each entry in the position column is the previous position value plus the change in position (velocity):

at t = 1, position = (previous position) + (change in position) = 5 + 6 = 11

at t = 2, position = (previous position) + (change in position) = 11 + 12 = 23

at t = 3, position = (previous position) + (change in position) = 23 + 16 = 39.

The rest of the entries in the position column are calculated in the same way.

Practice 4: Fig. 13 shows the initial acceleration, velocity and position of an object along the y–axis as well as its acceleration at 1 second time intervals. Fill in the empty spaces in the table and determine the position of the object on the y–axis after 9 seconds?

time (sec)	acceleration (ft/sec)	velocity (ft/sec)	position (ft)
0	0	2	5
1	4		
2	6		
3	8		
4	–3		
5	0		
6	5		
7	–2		
8	1		
9	0		

Fig. 13

PROBLEMS

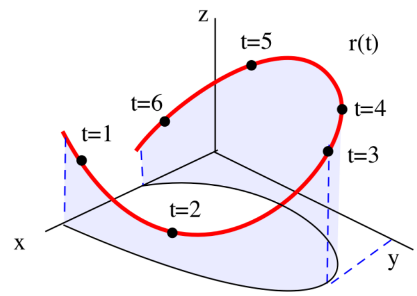

Fig. 14

In problems 1 – 4, fill in each component of $\mathbf{r}'$ with " + ", " – ", or " 0 ."

1. For $\mathbf{r}(t)$ in Fig. 14, $\mathbf{r}'(1) = \langle \ , \ , \ \rangle$, $\mathbf{r}'(2) = \langle \ , \ , \ \rangle$, and $\mathbf{r}'(3) = \langle \ , \ , \ \rangle$.

2. For $\mathbf{r}(t)$ in Fig. 14, $\mathbf{r}'(4) = \langle \ , \ , \ \rangle$, $\mathbf{r}'(5) = \langle \ , \ , \ \rangle$, and $\mathbf{r}'(6) = \langle \ , \ , \ \rangle$.

3. For $\mathbf{r}(t)$ in Fig. 15, $\mathbf{r}'(1) = \langle \ , \ , \ \rangle$, $\mathbf{r}'(2) = \langle \ , \ , \ \rangle$, and $\mathbf{r}'(3) = \langle \ , \ , \ \rangle$.

4. For $\mathbf{r}(t)$ in Fig. 15, $\mathbf{r}'(4) = \langle \ , \ , \ \rangle$, $\mathbf{r}'(5) = \langle \ , \ , \ \rangle$, and $\mathbf{r}'(6) = \langle \ , \ , \ \rangle$.

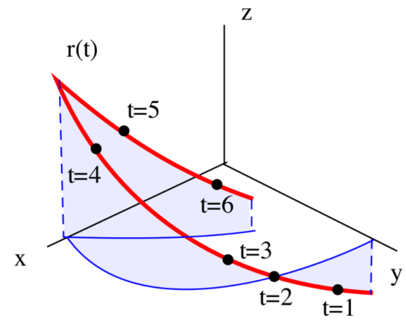

Fig. 15

In problems 5 – 8, the position vector $\mathbf{r}(t)$ is given for an object at time t. Calculate the velocity, speed, direction, and acceleration of the object at the given times.

5. $\mathbf{r}(t) = \langle t^3, 3+2t, t^2 \rangle$ and t = 1 and 2.

6. $\mathbf{r}(t) = \langle 5+3t^2, \sqrt{t}, t-t^3 \rangle$ and t = 1 and 2.

7. $\mathbf{r}(t) = (2-t)\mathbf{i} + (4/t)\mathbf{j} + (3)\mathbf{k}$ and t = 1 and 2.

8. $\mathbf{r}(t) = (2-t^3)\mathbf{i} + (5t)\mathbf{j} + (3+t)\mathbf{k}$ and t = 1 and 2.

9. $\mathbf{r}(t) = \langle t^3, 7, 1+5t \rangle$. Calculate $\frac{d}{dt}\mathbf{r}(2t)$.

10. $\mathbf{r}(t) = \langle 1/t, 6+5t, t^3 \rangle$. Calculate $\frac{d}{dt}\mathbf{r}(t^2)$.

11. $\mathbf{r}(t) = \langle t, 2t^2, 3t^3 \rangle$. Calculate $\frac{d}{dt}\{\sin(t)\,\mathbf{r}(t)\}$.

12. $\mathbf{r}(t) = \langle 7-t^2, 4, t^3-t \rangle$. Calculate $\frac{d}{dt}\{t^3\,\mathbf{r}(t)\}$.

13. $\mathbf{r}(t) = (2-5t^3)\mathbf{i} + (7t)\mathbf{j} + (1+t)\mathbf{k}$. Calculate $\frac{d}{dt}\mathbf{r}(3t)$.

14. $\mathbf{r}(t) = (1-t^2)\mathbf{i} + (5t^3)\mathbf{j} + (3+2t)\mathbf{k}$. Calculate $\frac{d}{dt}\mathbf{r}(t^3)$.

In problems 15 – 18, determine $\frac{d}{dt}\{\mathbf{u}+2\mathbf{v}\}$, $\frac{d}{dt}\{\mathbf{u}\cdot\mathbf{v}\}$, and $\frac{d}{dt}\{\mathbf{u}\times\mathbf{v}\}$ for the given vectors $\mathbf{u}(t)$ and $\mathbf{v}(t)$.

15. $\mathbf{u}(t) = \langle 0, t, t^3 \rangle$ and $\mathbf{v}(t) = \langle 1+5t, 4-t, 3 \rangle$.

16. $\mathbf{u}(t) = \langle 4t, 1, 5-t \rangle$ and $\mathbf{v}(t) = \langle t^2, 2+3t, t \rangle$

17. $\mathbf{u}(t) = (5t^3)\mathbf{i} + (2-7t)\mathbf{j} + (t+2)\mathbf{k}$ and $\mathbf{v}(t) = (1-2t)\mathbf{i} + (3t)\mathbf{j} + (4)\mathbf{k}$

18. $\mathbf{u}(t) = (2t)\mathbf{i} + (4)\mathbf{k}$ and $\mathbf{v}(t) = (1)\mathbf{i} + (2t)\mathbf{j} + (3t^2)\mathbf{k}$

In problems 19 – 22, find the point and angle of intersection for the given curves.

19. $\mathbf{u}(t) = \langle 3 - t , t , t^2 \rangle$ and $\mathbf{v}(t) = \langle 0 , t , 9 \rangle$

20. $\mathbf{u}(t) = \langle 4 - t^2 , t , t^2 \rangle$ and $\mathbf{v}(t) = \langle 3 , t^2 , \sqrt{t} \rangle$

21. $\mathbf{u}(t) = (5t^2)\mathbf{i} + (9)\mathbf{j} + (2 - t)\mathbf{k}$ and $\mathbf{v}(s) = (2 + s)\mathbf{i} + (3s)\mathbf{j} + (6 - s)\mathbf{k}$

22. $\mathbf{u}(t) = (2 + t)\mathbf{i} + (7 - t)\mathbf{j} + (t + 4)\mathbf{k}$ and $\mathbf{v}(s) = (3s)\mathbf{i} + (s + 1)\mathbf{j} + (s^3)\mathbf{k}$

23. The vectors $\mathbf{u}(t) = \langle 0 , t , t^2 \rangle$ and $\mathbf{v}(t) = \langle t , 2t , 0 \rangle$ form two sides of a parallelogram. How fast is the area of the parallelogram changing when $t = 1$. When $t = 2$?

24. The vectors $\mathbf{r}(t) = \langle 2t , 1 , 0 \rangle$ and $\mathbf{s}(t) = \langle 1 , 0 , 3 \rangle$ form two sides of a parallelogram. How fast is the area of the parallelogram changing when $t = 1$. When $t = 2$?

25. The vectors $\mathbf{u}(t) = \langle 1 , t , 3 \rangle$ and $\mathbf{v}(t) = \langle 2t , 0 , 0 \rangle$ form two sides of a triangle. How fast is the area of the triangle changing when $t = 1$. When $t = 2$?

26. The vectors $\mathbf{r}(t) = \langle t^2 , t , 1 \rangle$ and $\mathbf{s}(t) = \langle t , t^2 , 0 \rangle$ form two sides of a triangle. How fast is the area of the triangle changing when $t = 1$. When $t = 2$?

27. The vectors $\mathbf{u}(t) = \langle 1 , 0 , 0 \rangle$, $\mathbf{v}(t) = \langle 1 , t , 0 \rangle$ and $\mathbf{s}(t) = \langle 0 , t , 3t \rangle$ form three sides of a tetrahedron. (a) How fast is the volume of the tetrahedron changing when $t = 1$. When $t = 2$?
(b) How fast is the surface area of the tetrahedron changing when $t = 1$. When $t = 2$?

28. The vectors $\mathbf{u}(t) = \langle 2t , 0 , 0 \rangle$, $\mathbf{v}(t) = \langle 0 , 3t , 0 \rangle$ and $\mathbf{s}(t) = \langle 0 , 0 , 4t \rangle$ form three sides of a tetrahedron. (a) How fast is the volume of the tetrahedron changing when $t = 1$. When $t = 2$?
(b) How fast is the surface area of the tetrahedron changing when $t = 1$. When $t = 2$?

In problems 29 – 32, use the given information to find a formula for $\mathbf{r}(t)$.

29. $\mathbf{r}'(t) = \langle 12t , 12t^2 , 6e^t \rangle$ and $\mathbf{r}(0) = \langle 1 , 2 , 3 \rangle$.

30. $\mathbf{r}'(t) = \langle 3 + 4t , \cos(t), 1 - 6t \rangle$ and $\mathbf{r}(0) = \langle 7 , 2 , 5 \rangle$.

31. $\mathbf{r}'(t) = (6t^2)\mathbf{i} + (4)\mathbf{j} + (8t - 5)\mathbf{k}$ and $\mathbf{r}(1) = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

32. $\mathbf{r}'(t) = (5t^2)\mathbf{i} + (8t)\mathbf{j} + (2 - t)\mathbf{k}$ and $\mathbf{r}(2) = (3)\mathbf{i} + (7)\mathbf{j} + (0)\mathbf{k}$.

33. Fill in the rest of the **i** coordinate entries for **r** and **r '** in the table in Fig. 16.

34. Fill in the rest of the **j** coordinate entries for **r** and **r '** in the table in Fig. 16.

35. Fill in the rest of the **i** coordinate entries for **r** and **r '** in the table in Fig. 17.

36. Fill in the rest of the **j** coordinate entries for **r** and **r '** in the table in Fig. 17.

37. State and prove a differentiation rule for $\frac{d}{dt}\left(\frac{\mathbf{u}(t)}{f(t)}\right)$.

38. Prove that $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = -\frac{d}{dt}(\mathbf{v}(t) \times \mathbf{u}(t))$.

t	**r ''** (t)	**r '** (t)	**r**(t)
0	⟨ 0, 2, 5 ⟩	⟨ 1 , 2 , 3 ⟩	⟨ 0 , 3 , 1 ⟩
1	⟨ 4, 1, 3 ⟩	⟨ , , 6 ⟩	⟨ , , 7 ⟩
2	⟨ 6, 0, 1 ⟩	⟨ , , 7 ⟩	⟨ , , 14 ⟩
3	⟨ 4, –2, 0 ⟩	⟨ , , 7 ⟩	⟨ , , 21 ⟩
4	⟨ 2, 0, 2 ⟩	⟨ , , 9 ⟩	⟨ , , 30 ⟩
5	⟨ 8, 3, 4 ⟩	⟨ , , 13 ⟩	⟨ , , 43 ⟩

Fig. 16

t	**r ''** (t)	**r '** (t)	**r**(t)
0	⟨ 1, 2, 3 ⟩	⟨ , , 4 ⟩	⟨ , , 2 ⟩
1	⟨ 4, 2, 2 ⟩	⟨ , , 6 ⟩	⟨ , , 8 ⟩
2	⟨ 3, 1, 0 ⟩	⟨ 8 , 9 , 6 ⟩	⟨ 30, 20 , 14 ⟩
3	⟨ 2, 3, 1 ⟩	⟨ , , 7 ⟩	⟨ , , 21 ⟩
4	⟨ 1, 4, 0 ⟩	⟨ , , 7 ⟩	⟨ , , 28 ⟩
5	⟨ 0, 1, 3 ⟩	⟨ , , 10 ⟩	⟨ , , 38 ⟩

Fig. 17

Practice Answers

Practice 1: $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. $\mathbf{v}(t) = \text{velocity} = \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$ so
$\mathbf{v}(0) = \langle 1, 0, 0 \rangle$, $\mathbf{v}(1) = \langle 1, 2, 3 \rangle$, $\mathbf{v}(2) = \langle 1, 4, 12 \rangle$.

$\mathbf{sp}(t) = \text{speed} = |\mathbf{v}(t)|$ so $\mathbf{sp}(0) = 1$, $\mathbf{sp}(1) = \sqrt{14}$, $\mathbf{sp}(2) = \sqrt{161}$.

$\mathbf{dir}(t) = \text{direction} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$. $\mathbf{dir}(0) = \langle 1, 0, 0 \rangle$, $\mathbf{dir}(1) = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle$, $\mathbf{dir}(2) = \frac{1}{\sqrt{161}} \langle 1, 4, 12 \rangle$.

$\mathbf{a}(t) = \text{acceleration} = \mathbf{r}''(t) = \langle 0, 2, 6t \rangle$ so $\mathbf{a}(0) = \langle 0, 2, 0 \rangle$, $\mathbf{a}(1) = \langle 0, 2, 6 \rangle$, $\mathbf{a}(2) = \langle 0, 2, 12 \rangle$.

Practice 2: The paths $\mathbf{r}(t) = \langle 0, t, t^2 \rangle$ and $\mathbf{s}(t) = \langle 2 - t, 1, 5 - t^2 \rangle$ intersect at $\mathbf{r}(1) = \mathbf{s}(2) = (0,1,1)$.
$\mathbf{r}'(t) = \langle 0, 1, 2t \rangle$ and $\mathbf{s}'(t) = \langle -1, 0, -2t \rangle$ so $\mathbf{r}'(1) = \langle 0, 1, 2 \rangle$ and $\mathbf{s}'(2) = \langle -1, 0, -4 \rangle$.

$\cos(\theta) = \frac{\mathbf{r}'(1) \cdot \mathbf{s}'(2)}{|\mathbf{r}'(1)||\mathbf{s}'(2)|} = \frac{-8}{\sqrt{5}\sqrt{17}} = -0.868$ so the angle between $\mathbf{r}'(1)$ and $\mathbf{s}'(1)$ is

$\theta \approx 2.922$ (or 150.2^{o})

Practice 3: $\mathbf{r}'(t) = \mathbf{v}(t) = \langle 6t^2, \cos(t), 12e^{3t} \rangle$ so $\mathbf{r}(t) = \langle 2t^3 + A, \sin(t) + B, 4e^{3t} + C \rangle$.
Since $\mathbf{r}(0) = \langle 1, -5, 2 \rangle = \langle 2(0)^3 + A, \sin(0) + B, 4e^{3(0)} + C \rangle = \langle A, B, 4 + C \rangle$, we have
$A = 1$, $B = -5$, and $C = -2$. Then $\mathbf{r}(t) = \langle 2t^3 + 1, \sin(t) - 5, 4e^{3t} - 2 \rangle$.

Practice 4: Velocity entries: 2, 6, 12, 20, 17, 17, 22, 20, 21, 21

Acceleration entries: 5, 11, 23, 43, 60, 77, 99, 119, 140, 161