12.1   VECTOR–VALUED FUNCTIONS AND
CURVES IN SPACE

As a bug buzzes around a room (Fig. 1) or a submarine explores the oceans

or a planet orbits a moving star, we could describe the location

of the bug or submarine or planet at any particular time as a point

(x, y, z) in 3–dimensional space. But to describe the object's movement

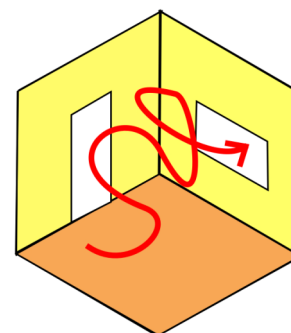we need the locations at many times, and that leads very naturally to the

idea of representing the path of the object as a set of

points $P(t) = (x(t), y(t), z(t))$ given by parametric

equations $x = x(t)$, $y = y(t)$, and $z = z(t)$ where the

variable t represents the parameter time. Fig. 2

shows the paths of $P(t) = (1 + t, 0 + 3t, 1 + 2t)$ and

$Q(t) = (\cos(t), \sin(t), t)$. Such parametric equations

and their graphs are fundamental to our study of

motion in 3–dimensional space, but for many uses it is

more effective to work with vectors rather than points,
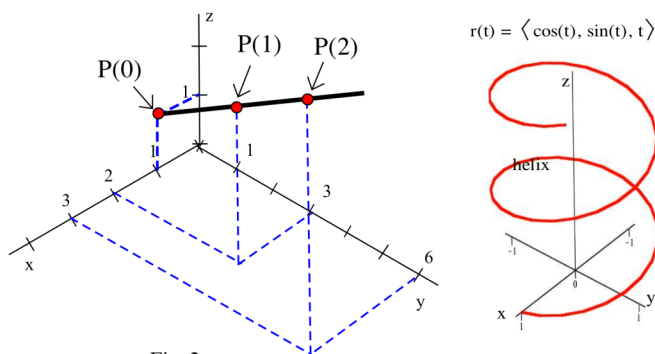
and that leads to **vector–valued** functions.

Fig. 1: Bug's path

Fig. 2

Definition: A **vector–valued function** is a rule that assigns a vector to each input number.

Typically, a vector–valued function has the form

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \langle\, x(t), y(t), z(t) \,\rangle$$

where x, y, and z are scalar–valued functions.

The domain of a vector–valued function $\mathbf{r}(t)$ is a set of real numbers: the domain of $\mathbf{r}$ consists
of those t in the domains of x, y, and z.

The range of a vector–valued function is a collection of vectors.

Vector–valued functions offer two advantages that should become clearer as you work with them. First,

vector–valued functions are often notationally simpler than parametric equations — it is easier to write

$\mathbf{r}(t)$ than $(x(t), y(t), z(t))$. This is not a big advantage; P(t) is also easy to write, but the $\mathbf{r}(t)$ notation does

make some ideas and computations easier. The second advantage of vector–valued functions is that they

allow us to use the powerful machinery we have already developed for working with vectors. This vector

machinery is particularly useful when we consider **tangent vectors** to curves and **angles** between curves.

Our discussion of vector–valued functions is similar to the discussion of functions of the form $y = f(x)$ in the early chapters. We begin by considering their graphs and the ideas of limits and continuity for vector–valued functions.

Graphs of Vector–Valued Functions — Space Curves

If an object is located at the point $P(t) = (x(t), y(t), z(t))$ at time t, then we say that

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \left\langle x(t), y(t), z(t) \right\rangle,$$

a vector from the origin to the point $P(t)$, is the object's **position vector** . The functions $x(t), y(t),$ and $z(t)$ are called the **components** of $\mathbf{r}(t)$ or the **component functions** of $\mathbf{r}(t)$. Fig. 3 shows a point $P(t)$ and the position vector $\mathbf{r}(t)$ for that point. It is difficult to draw and difficult to interpret a collection of vectors, so typically when we work with the graphs of vector–valued functions we draw only the path of the endpoints of the vectors: the graph of $\mathbf{r}(t) = \left\langle x(t), y(t), z(t) \right\rangle$ is the collection of points $P(t) = (x(t), y(t), z(t))$.

Fig. 3
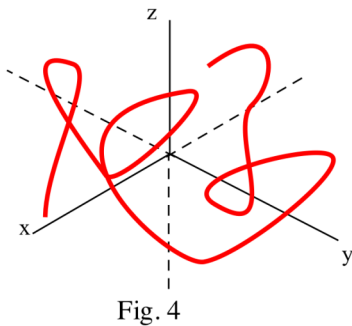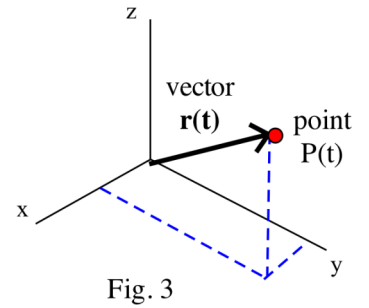
Fig. 4

The graphs of vector–valued functions can be very complex and difficult to sketch, for example Fig. 4, but many are manageable "by hand" and you should practice sketching some of their graphs.

Example 1:   Sketch the graphs of $\mathbf{r}(t) = \left\langle 3 - t, 2t, 4 - 2t \right\rangle$ for $0 \le t \le 3$, and $\mathbf{s}(t) = \left\langle t, 0, \sin(t) \right\rangle$ for $0 \le t \le 2\pi$.

Solution: The graphs are shown in Fig. 5.

Fig. 5
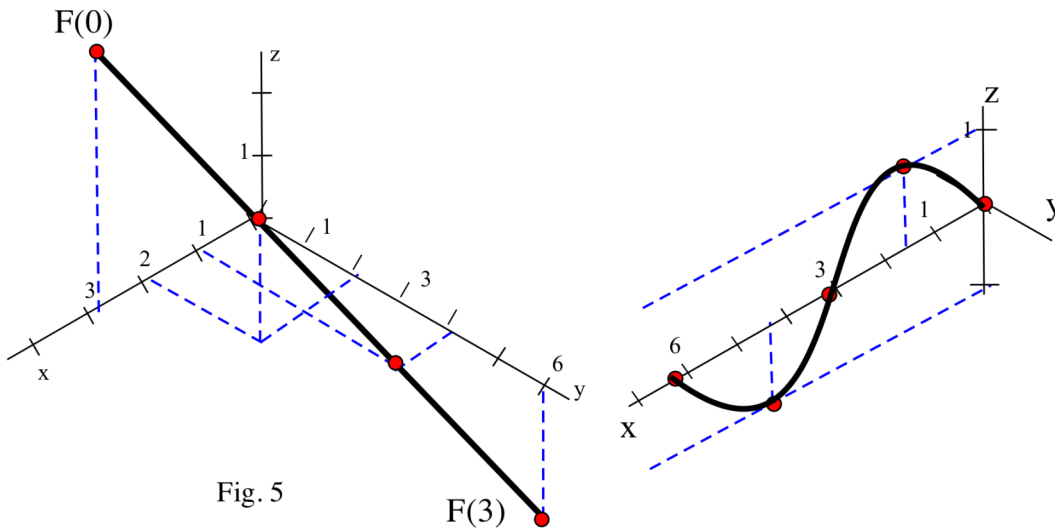
Practice 1:   Sketch the graphs of $\mathbf{r}(t) = \langle\, t, 4 - 2t, 2 - t\,\rangle$ for $0 \le t \le 2$, and $\mathbf{s}(t) = \langle\, t, t^2, 1\,\rangle$ for $0 \le t \le 2$.

Sometimes the component functions $x(t), y(t),$ and $z(t)$ may only be given as graphic information, but we can still sketch the graph of the vector–valued function they define.

Example 2:   The graphs of $x(t), y(t),$ and $z(t)$ are shown in Fig. 6. Use this information about the component functions of $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
$= \langle\, x(t), y(t), z(t)\,\rangle$
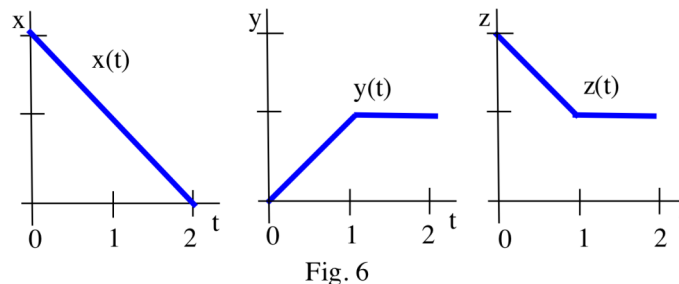to sketch a graph of $\mathbf{r}(t)$.

Fig. 6

Solution: The graph of $\mathbf{r}(t)$ is shown in Fig. 7.

Practice 2:   The graphs of $x(t), y(t),$ and $z(t)$ are shown in Fig. 8.

Use this information about the component functions of
$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \langle\, x(t), y(t), z(t)\,\rangle$ to sketch a graph of $\mathbf{r}(t)$.
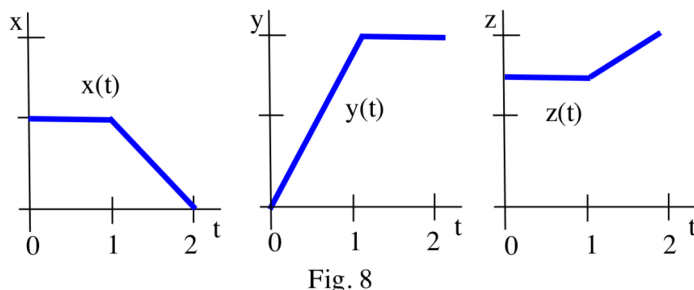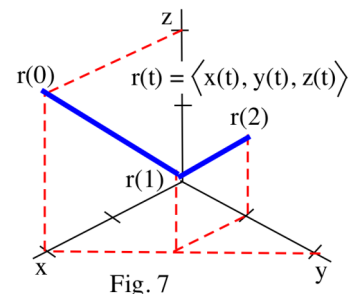

Fig. 8

Fig. 7

Limits and Continuity of Vector–Valued Functions

Limits and continuity of vector–valued functions are defined in terms of the components of the function. If you understand the meaning of limits and continuity for functions $y = f(x)$, then these concepts are not difficult for vector–valued functions.

Definition: For $\mathbf{r}(t) = \langle\, x(t), y(t), z(t)\,\rangle$,

$$\lim_{t \to a} \mathbf{r}(t) = \langle\, \lim_{t \to a} x(t)\,, \lim_{t \to a} y(t)\,, \lim_{t \to a} z(t)\,\rangle$$

provided that each of the limits of the component functions exists.

To calculate the limit of a vector–valued function we simply need the limits of each of the component scalar–valued functions. If any of the limits of the component functions fail to exist, we say that the limit of the vector–valued function does not exist. The various properties of limits of vector–valued functions follow directly from the corresponding properties of scalar–valued functions as they are applied to each component separately.

Example 3:   Determine $\lim\limits_{t \to 0} \mathbf{r}(t)$ and $\lim\limits_{t \to \infty} \mathbf{s}(t)$ for $\mathbf{r}(t) = \left\langle \cos(t) , \sqrt{4 + t} , 3e^{2t} \right\rangle$

and $\mathbf{s}(t) = \left\langle \frac{2t}{t+3} , \frac{5}{t} , 3 + \frac{\sin(t)}{2t} \right\rangle$

Solution: $\lim\limits_{t \to 0} \mathbf{r}(t) = \left\langle \lim\limits_{t \to 0} \cos(t) , \lim\limits_{t \to 0} \sqrt{4 + t} , \lim\limits_{t \to 0} 3e^{2t} \right\rangle = \left\langle 1, 2, 3 \right\rangle$ and

$\lim\limits_{t \to \infty} \mathbf{s}(t) = \left\langle \lim\limits_{t \to \infty} \frac{2t}{t + 3} , \lim\limits_{t \to \infty} \frac{5}{t} , \lim\limits_{t \to \infty} 3 + \frac{\sin(t)}{2t} \right\rangle = \left\langle 2, 0, 3 \right\rangle$

Practice 3:   Determine $\lim\limits_{t \to \pi} \mathbf{r}(t)$ and $\lim\limits_{t \to \infty} \mathbf{r}(t)$ for $\mathbf{r}(t) = \left\langle \cos(t) , \frac{\sin(t)}{t} , 3 \right\rangle$

Continuity of vector–valued functions is stated in terms of limits, so the continuity of a vector–valued function depends on the continuity (and limits) of the component functions.

Definition: A vector–valued function $\mathbf{r}(t) = \left\langle x(t), y(t), z(t) \right\rangle$ is

continuous at the point $t = t_0$

if $\lim\limits_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$.

The following result about continuity follows directly from the definition and is usually easier to use.

Component Continuity Theorem for Vector–valued Functions

A vector–valued function $\mathbf{r}(t) = \left\langle x(t), y(t), z(t) \right\rangle$ is continuous at the point $t = t_0$

if and only if each of the component functions $x(t)$, $y(t)$, and $z(t)$ is continuous at $t = t_0$.

Proof: The proof follows directly from the definitions of continuity and limits of vector–valued functions.
          If $\mathbf{r}(t)$ is continuous at $t = t_0$, then

$$\lim\limits_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0) = \left\langle x(t_0), y(t_0), z(t_0) \right\rangle .$$

But, from the definition of limit of $\mathbf{r}(t)$, $\lim\limits_{t \to a} \mathbf{r}(t) = \left\langle \lim\limits_{t \to a} x(t) , \lim\limits_{t \to a} y(t) , \lim\limits_{t \to a} z(t) \right\rangle .$

Two vectors are equal if and only if their respective components are equal, so

$$\lim_{t \to t_0} x(t) = x(t_0) \ , \ \lim_{t \to t_0} y(t) = y(t_0) \ , \text{ and } \ \lim_{t \to t_0} z(t) = z(t_0) \ ,$$

and we have shown that x, y, and z are continuous at $t = t_0$.

The proof that "if $x(t), y(t),$ and $z(t)$ are continuous at $t = t_0$, then $\mathbf{r}(t)$ is continuous at $t = t_0$" is

similar but starts with the assumption that x, y, and z are continuous at $t = t_0$.

Example 4:   Where are $\mathbf{r}(t) = \left\langle \cos(t), \sin(t), t^2 \right\rangle$ and $\mathbf{s}(t) = \left\langle 2 + t, \frac{1}{t-3} \ , \ln(t) \right\rangle$ continuous?

Solution: (a) $\mathbf{r}(t) = \left\langle \cos(t), \sin(t), t^2 \right\rangle$ is continuous everywhere (for all t) since all of the component

functions $x(t) = \cos(t), y(t) = \sin(t),$ and $z(t) = t^2$ are continuous for all values of t.

(b) $\mathbf{s}(t) = \left\langle 2 + t, \frac{1}{t-3} \ , \ln(t) \right\rangle$ is continuous for $0 < t < 3$ and $3 < t$ since $x(t) = 2 + t$ is continuous for

all values of t; $y(t) = \frac{1}{t-3}$ is continuous for $t \neq 3$; and $z(t) = \ln(t)$ is continuous for $t > 0$.

Practice 4:   Where is $\mathbf{r}(t) = \left\langle 1/t \ , e^t, \text{INT}(t) \right\rangle$ continuous?

Fig. 9 shows the graphs of continuous components x(t) and y(t) and a discontinuous component z(t). It also

shows the discontinuous vector–valued function $\mathbf{r}(t) = \left\langle x(t) \ , y(t), z(t) \right\rangle$.
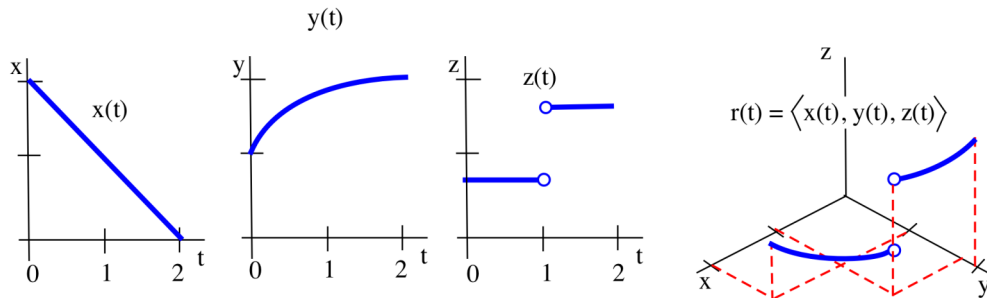

Fig. 9

Some Useful Space Curves: Lines, Helix, and Bezier Curves

By this point in your mathematical development, the graphs of some functions should be almost "automatic."

You should be able to visualize the graphs of $y = x^2, y = 3 + 2\sin(x), y = |x-2|$ and a variety of others with

little effort. It is useful to **start** to develop similar skills with the graphs of a few

vector–valued functions. Lines and helices are useful shapes to begin with, and Bezier curves in three

dimensions have a number of useful properties.

Lines

If $\mathbf{r}(t) = \langle\, x(t), y(t), z(t)\,\rangle$, and each of the component functions $x(t), y(t),$ and $z(t)$ is a linear function

$(at + b)$, then the graph of $\mathbf{r}(t)$ is a straight line in space. In this case we need only evaluate and plot $\mathbf{r}(t)$

for a couple values of t before finishing the graph with a straightedge.

Example 5:   Sketch the graphs of $\mathbf{r}(t) = \langle\, 2 + t, 3 - t, 1 + 2t\,\rangle$ and $\mathbf{s}(t) = \langle\, 2, t, 3\,\rangle$.

Solution: $\mathbf{r}(0) = \langle\, 2, 3, 1\,\rangle$ and $\mathbf{r}(1) = \langle\, 3, 2, 3\,\rangle$. Fig. 10 shows these two points and a graph of $\mathbf{r}(t)$.

$\qquad\quad$ $\mathbf{s}(0) = \langle\, 2, 0, 3\,\rangle$ and $\mathbf{s}(1) = \langle\, 2, 1, 3\,\rangle$. Fig. 11 shows these two points and a graph of $\mathbf{s}(t)$.
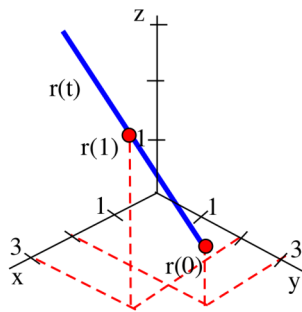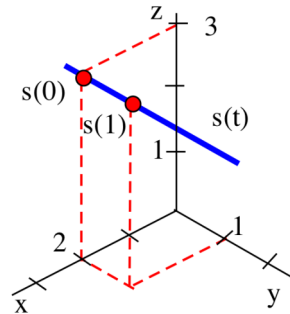
Fig. 10   Fig. 11

Practice 5:   Sketch the graphs of $\mathbf{r}(t) = \langle\, t - 1, 0, 1 + 2t\,\rangle$ and $\mathbf{s}(t) = \langle\, 2, 4, 3t\,\rangle$.

If $x(t), y(t),$ and $z(t)$ are linear functions, then they are continuous for all values of t so
$\mathbf{r}(t) = \langle\, x(t), y(t), z(t)\,\rangle$ is also continuous for all values of t (by the Component Continuity Theorem).

The Helix and Some Variations

The graph of $\mathbf{r}(t) = \langle\, \cos(t), \sin(t), t\,\rangle$ is shown in Fig. 12, and it is called a

helix, or a **circular helix around the z–axis**. The parametric graph of

$(\,\cos(t), \sin(t)\,)$ is a circle in the xy–plane (Fig. 13), and $z(t) = t$ then "stretches"

this circle into the z direction to create the spiral shape (Fig. 12). The helix is

sometimes useful, and it provides a focus to investigate the effects on the shape of

the graph of certain changes in the component functions.

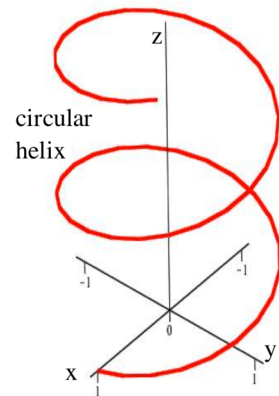$\mathbf{r}(t) = \langle\, \cos(t), \sin(t), t\,\rangle$

circular
helix

Fig. 12

$P(t) = (\,\cos(t), \sin(t)\,)$
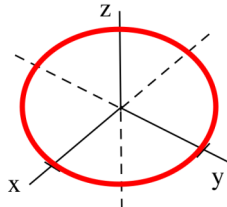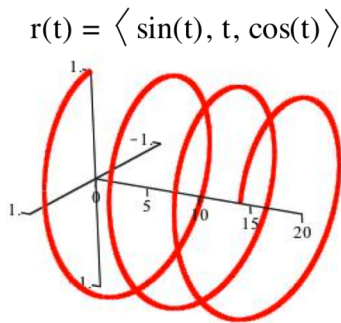a circle in the xy-plane

Fig. 13

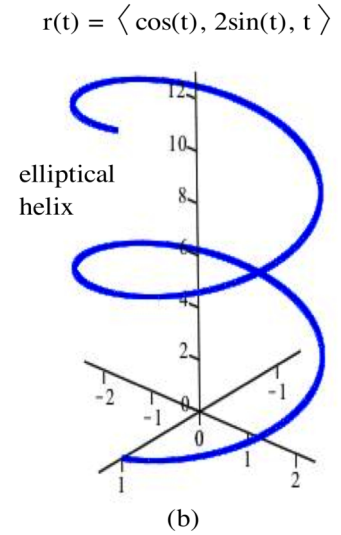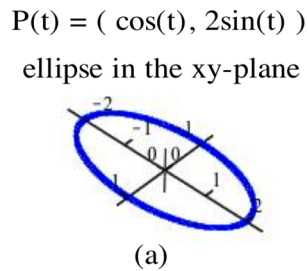**Example 6**:   (a) Without plotting any points, describe the

shapes of the graphs of $\mathbf{r}(t) = \langle\, \cos(t), t, \sin(t) \,\rangle$ and $\mathbf{s}(t) = \langle\, \cos(t), 2\cdot\sin(t), t \,\rangle$.

(b) Then sketch the graphs of $\mathbf{r}(t)$ and $\mathbf{s}(t)$.

Solution: (a) The graph of $\mathbf{r}(t) = \langle\, \sin(t), t, \cos(t) \,\rangle$ is a circular helix around the y–axis (Fig. 14)

(b) The parametric graph of $(\, \cos(t), 2\cdot\sin(t) \,)$ in the xy–plane is an ellipse (Fig. 15a), so the

graph of $\mathbf{s}(t) = \langle\, \cos(t), 2\cdot\sin(t), t \,\rangle$ is an elliptical helix around the z–axis (Fig. 15b)

$\mathbf{r}(t) = \langle\, \sin(t), t, \cos(t) \,\rangle$

Fig. 14

$P(t) = (\, \cos(t), 2\sin(t) \,)$

ellipse in the xy-plane

(a)

Fig. 15

$\mathbf{r}(t) = \langle\, \cos(t), 2\sin(t), t \,\rangle$

elliptical helix

(b)

Practice 6:   (a) Without plotting any points, describe the shapes of the

graphs of $\mathbf{r}(t) = \langle\, t, \cos(t), \sin(t) \,\rangle$ and $\mathbf{s}(t) = \langle\, 2\cdot\cos(t), t, 3\cdot\sin(t) \,\rangle$.

(b) Then sketch the graphs of $\mathbf{r}(t)$ and $\mathbf{s}(t)$.

Figs. 16 and 17 show the graphs of two more variations on the component functions of the original helix.

$\mathbf{r}(t) = \langle\, \cos(t), \sin(t), t^2 \,\rangle$

Fig. 16

$\mathbf{r}(t) = \langle\, t\cos(t), t\sin(t), t \,\rangle$
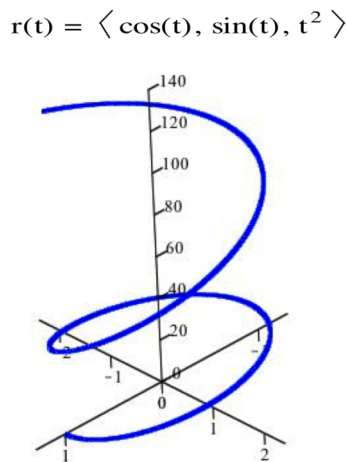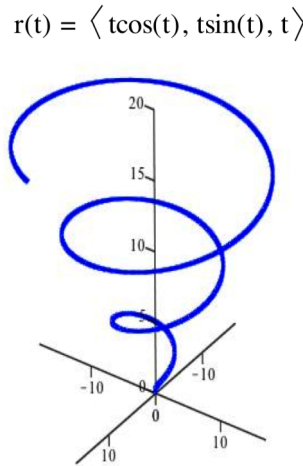
Fig. 17

Bezier Curves in Three Dimensions

One powerful aspect of the Bezier curves we saw in Section 9.4.5 is that the ideas and even the formulas for these curves extend very easily to Bezier curves in three dimensions. In Section 9.4.5 we saw that if we start with four points $P_0 = (x_0, y_0), P_1 = (x_1, y_1), P_2 = (x_2, y_2),$ and $P_3 = (x_3, y_3)$ then, for $0 \leq t \leq 1,$ the curve given by

Fig. 18

$$x(t) = (1-t)^3 \cdot x_0 + 3(1-t)^2 t \cdot x_1 + 3(1-t)t^2 \cdot x_2 + t^3 \cdot x_3 \quad \text{and}$$
$$y(t) = (1-t)^3 \cdot y_0 + 3(1-t)^2 t \cdot y_1 + 3(1-t)t^2 \cdot y_2 + t^3 \cdot y_3$$

(or more simply as $B(t) = (1-t)^3 \cdot P_0 + 3(1-t)^2 t \cdot P_1 + 3(1-t)t^2 \cdot P_2 + t^3 \cdot P_3$)

has several useful properties:

(1) $B(0) = P_0$ and $B(1) = P_3$ so P_0 and P_3 are the "endpoints" of $B(t)$ for $0 \leq t \leq 1$.

(2) $B(t)$ is a cubic polynomial so it is continuous and differentiable.

(3) $B'(0) = $ slope of the line segment from P_0 to P_1:
$B'(1) = $ slope of the line segment from P_2 to P_3.

(4) For $0 \leq t \leq 1$, the graph of $B(t)$ is in the region whose corners are the control points.

A Bezier curve $B(t)$ for four given points P_0, P_1, P_2 and P_3 is shown in Fig. 18 .

This extends very simply to three dimensions.
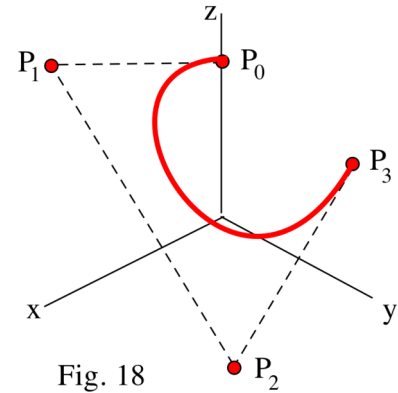
Bezier Curve in Three Dimensions

If P_0, P_1, P_2 and P_3 are four points in 3–dimensional space, then the Bezier curve in three dimensions for those points is

$$B(t) = (1-t)^3 \cdot P_0 + 3(1-t)^2 t \cdot P_1 + 3(1-t)t^2 \cdot P_2 + t^3 \cdot P_3 \quad \text{for } 0 \leq t \leq 1:$$

$$x(t) = (1-t)^3 \cdot x_0 + 3(1-t)^2 t \cdot x_1 + 3(1-t)t^2 \cdot x_2 + t^3 \cdot x_3 \,,$$
$$y(t) = (1-t)^3 \cdot y_0 + 3(1-t)^2 t \cdot y_1 + 3(1-t)t^2 \cdot y_2 + t^3 \cdot y_3 \,, \quad \text{and}$$
$$z(t) = (1-t)^3 \cdot z_0 + 3(1-t)^2 t \cdot z_1 + 3(1-t)t^2 \cdot z_2 + t^3 \cdot z_3 \,.$$

This 3–dimensional Bezier curve has the same properties as those listed for the
2–dimensional Bezier curve with two modifications.

Fig. 19

(3') When $t = 0$ the direction of B(t) is the same as the direction of the
line segment from P_0 to P_1

(Fig. 19): when $t = 1$ the direction of B(t) is the same as the direction of the
line segment from P_2 to P_3 . (We will define the "direction" of a space curve
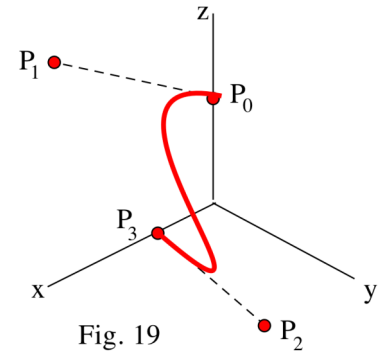in Section 12.2, but it is similar to "slope" in two dimensions.)

(4') For $0 \le t \le 1$, the graph of B(t) is "on
the rubber sheet" whose corners are the
control points (Fig. 20). (This is not very
precise, but it should convey the idea
without the need for more technical vocabulary.)
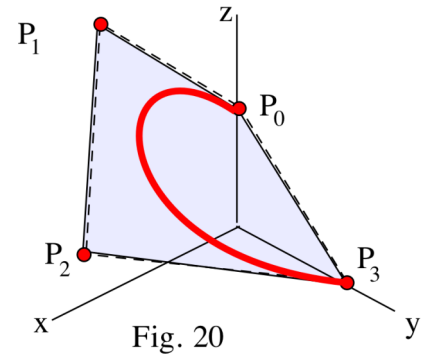
Fig. 20

PROBLEMS

In practice, most 3–dimensional graphs are created using computers. The
point of many of the following problems is to help develop your 3–
dimensional visualization skills.

Fig. 21

In problems 1 – 10, sketch the graph of each vector–valued function. Many
of these graphs lie on a coordinate plane so it is not difficult to sketch
relatively good graphs by hand.

1. Sketch $\mathbf{r}(t) = \left\langle t, t^2, 0 \right\rangle$ on the axes system in Fig. 21 for $0 \le t \le 2$.

2. Sketch $\mathbf{s}(t) = \left\langle t, 0, t^2 \right\rangle$ on the axes system in Fig. 21 for $0 \le t \le 2$.

3. Sketch $\mathbf{r}(t) = \left\langle 0, t^2, t \right\rangle$ on the axes   system in Fig. 22 for $0 \le t \le 2$.

4. Sketch $\mathbf{s}(t) = \left\langle t, 2t, 0 \right\rangle$ on the axes   system in Fig. 22 for $0 \le t \le 3$.
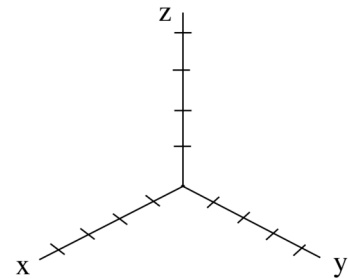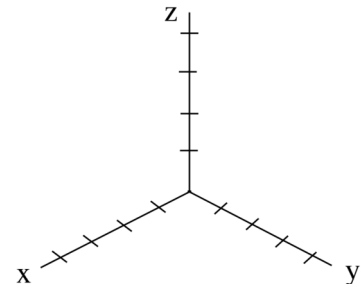
Fig. 22

5. Sketch $\mathbf{r}(t) = \langle 3t, 0, t \rangle$ on the axes system in Fig. 23 for $0 \le t \le 3$.

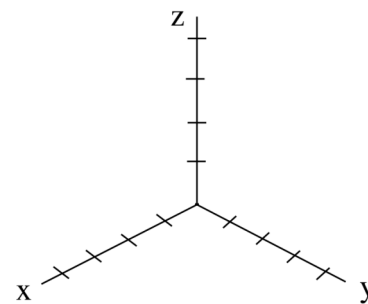6. Sketch $\mathbf{s}(t) = \langle 0, t, \sin(t) \rangle$ on the axes system in Fig. 23 for $0 \le t \le 2\pi$.

Fig. 23

7. Sketch $\mathbf{r}(t) = \langle 0, \sin(t), t \rangle$ on the axes system in Fig. 24 for $0 \le t \le 2\pi$.

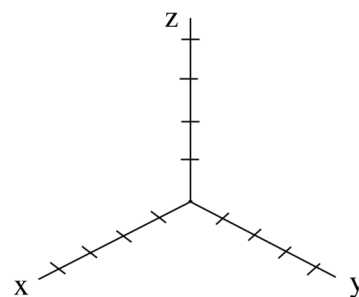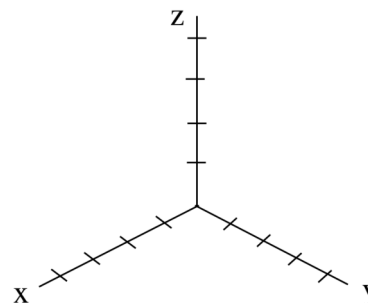8. Sketch $\mathbf{s}(t) = \langle \cos(t), \sin(t), 1 \rangle$ on the axes system in Fig. 24 for $0 \le t \le 2\pi$.

Fig. 24

9. Sketch $\mathbf{r}(t) = \langle 2, \sin(t), \cos(t) \rangle$ on the axes system in Fig. 25 for $0 \le t \le 2\pi$.

10. Sketch $\mathbf{s}(t) = \langle 0, \sin(t), \cos(t) \rangle$ on the axes system in Fig. 25 for $0 \le t \le 2\pi$.

Fig. 25

In problems 11 – 16, calculate and carefully plot three points on each vector–valued linear function and then complete the graph of the line.
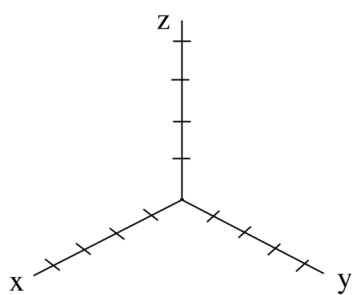
11. $\mathbf{r}(t) = \langle\, 2 - t, t, 1 + 2t \,\rangle$ in Fig. 26.

12. $\mathbf{s}(t) = \langle\, 2\ , t, 1 + 2t \,\rangle$ in Fig. 26.

13. $\mathbf{r}(t) = \langle\, t, 2, 3 \,\rangle$ in Fig. 27.

14. $\mathbf{s}(t) = \langle\, 1\ , t, 3 \,\rangle$ in Fig. 27.

15. $\mathbf{r}(t) = \langle\, t + 1, t - 2, 2t \,\rangle$ in Fig. 28.

16. $\mathbf{s}(t) = \langle\, 1 + 2t\ , t, 3 \,\rangle$ in Fig. 28.

Fig. 26

Fig. 27
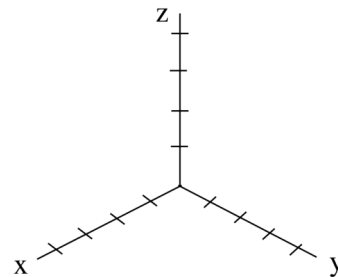
In problems 17 – 20, graphs are given for $x(t), y(t),$ and $z(t)$. Use the information in these graphs to graph the vector–valued function $\mathbf{r}(t) = \langle\, x(t), y(t), z(t) \,\rangle$ on the given coordinate system.

17. The graphs of $x(t), y(t),$ and $z(t)$ are in Fig. 29.

18. The graphs of $x(t), y(t),$ and $z(t)$ are in Fig. 30.
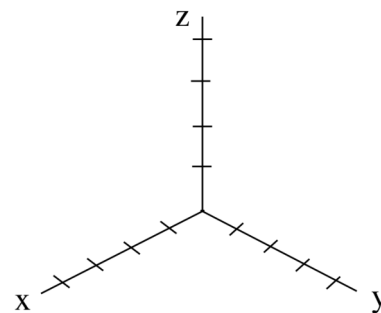
Fig. 28
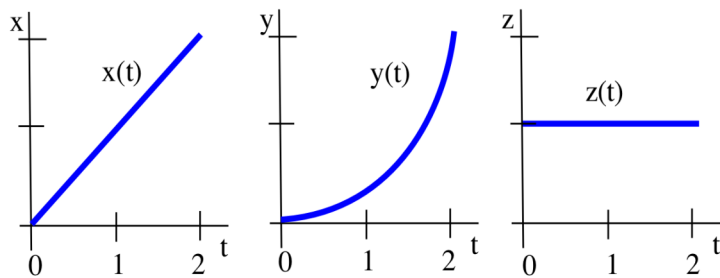
Fig. 29
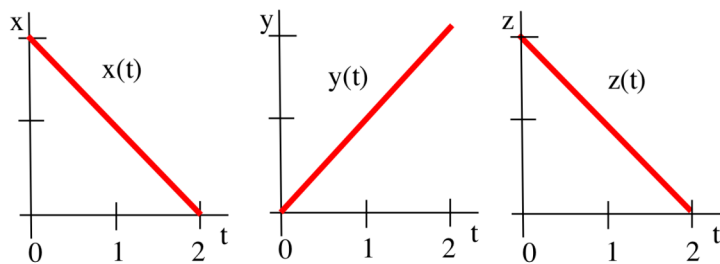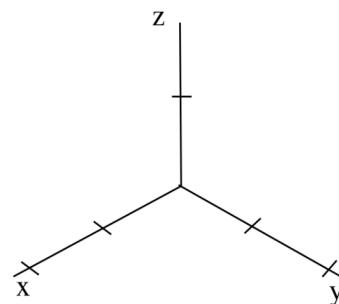
Fig. 30
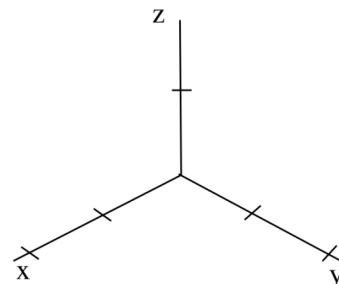
19. The graphs of x(t), y(t), and z(t) are in Fig. 31.   20. The graphs of x(t), y(t), and z(t) are in Fig. 32.
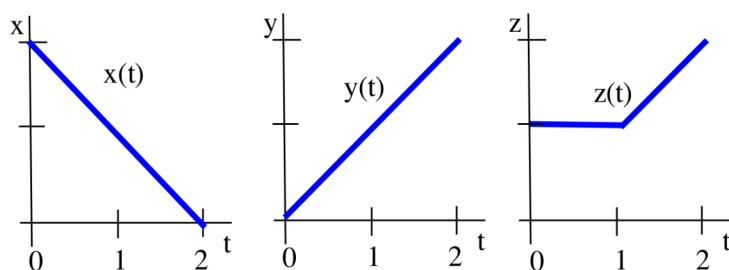In problems 21 – 28, calculate the limits.

Fig. 31
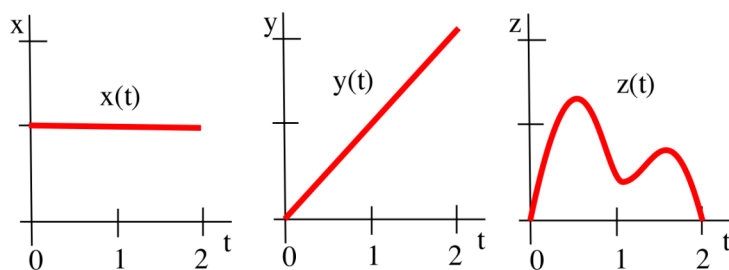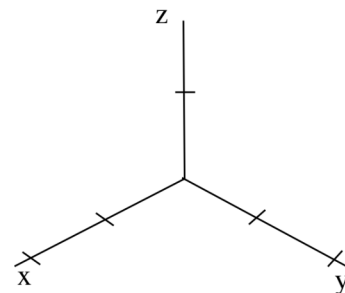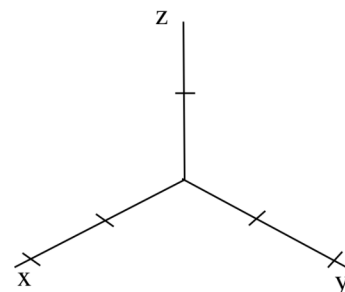

Fig. 32

21. $\lim_{t \to 3} \left\langle t , \dfrac{t^2}{2} , \dfrac{1}{t-1} \right\rangle$.

22. $\lim_{t \to 0} \left\langle \cos(t) , \dfrac{t}{3-t} , t^2 \right\rangle$.

23. $\lim_{t \to 2} \left\langle 4 , \dfrac{t}{t-2} , \sqrt{7+t} \right\rangle$.

24. $\lim_{t \to 2} \left\langle \sqrt{t-1} , t^3 , 1 \right\rangle$.

25. $\lim_{t \to \infty} \left\langle \dfrac{4}{t+1} , \dfrac{3t+1}{t-2} , \dfrac{\sin(t)}{t} \right\rangle$.

26. $\lim_{t \to \infty} \left\langle \dfrac{5t^2 + 2t + 1}{t^2 + t - 6} , \sin(t) , \dfrac{1}{1+t^2} \right\rangle$.

27. $\lim_{t \to \infty} \left\langle \arctan(t) , \dfrac{\ln(t)}{t} , \dfrac{3}{t^2} \right\rangle$.

28. $\lim_{t \to \infty} \left\langle \sin(\arctan(t)) , 1 , 5^{1/t} \right\rangle$.

In problems 29 – 36, determine where the given vector–valued functions are continuous.

29. $\mathbf{r}(t) = \left\langle t , \dfrac{t^2}{2} , \dfrac{1}{t-1} \right\rangle$.

30. $\mathbf{s}(t) = \left\langle \cos(t) , \dfrac{t}{3-t} , t^2 \right\rangle$.

31. $\mathbf{u}(t) = \left\langle 4 , \dfrac{t}{t-2} , \sqrt{7+t} \right\rangle$.

32. $\mathbf{v}(t) = \left\langle \sqrt{t-1} , t^3 , 1 \right\rangle$.

33. $\mathbf{w}(t) = \left\langle \dfrac{4}{t+1} , \dfrac{3t+1}{t-2} , \dfrac{\sin(t)}{t} \right\rangle$.

34. $\mathbf{r}(t) = \left\langle \dfrac{5t^2 + 2t + 1}{t^2 + t - 6} , \sin(t) , \dfrac{1}{1+t^2} \right\rangle$.

35. $\mathbf{r}(t) = \left\langle \arctan(t) , \dfrac{\ln(t)}{t} , \dfrac{3}{t^2} \right\rangle$.

36. $\mathbf{s}(t) = \left\langle \sin(\arctan(t)) , 1 , 5^{1/t} \right\rangle$.
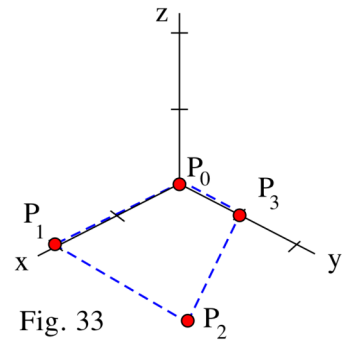
In problems 37 – 40, determine the Bezier curve B(t) for the given control points P_0, P_1, P_2 and P_3 . If you have access to a computer with the appropriate software, graph B(t).

37. $P_0 = (0, 0, 1), P_1 = (1, 0, 1), P_2 = (1, 2, 0),$ and $P_3 = (0, 2, 0)$.

38. $P_0 = (0, 0, 2), P_1 = (1, 0, 1), P_2 = (0, 2, 0),$ and $P_3 = (1, 3, 0)$.

39. $P_0 = (0, 1, 2), P_1 = (0, 0, 2), P_2 = (2, 1, 0),$ and $P_3 = (2, 0, 0)$.

40. $P_0 = (0, 1, 2), P_1 = (0, 0, 2), P_2 = (2, 0, 0),$ and $P_3 = (1, 1, 0)$.

In problems 41 – 44, control points P_0, P_1, P_2 and P_3 are shown on a graph. For the given control points sketch a curve with the properties of the Bezier curve. (In some graphs additional lines are included with the control points to help with your sketch.)

Fig. 33

41. P_0, P_1, P_2 and P_3 are given in Fig. 33.

42. P_0, P_1, P_2 and P_3 are given in Fig. 34.

43. P_0, P_1, P_2 and P_3 are given in Fig. 35.

44. P_0, P_1, P_2 and P_3 are given in Fig. 36.

g. 34

Fig. 35

Fig. 36
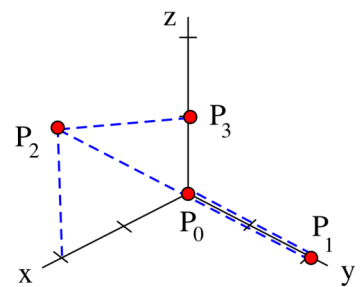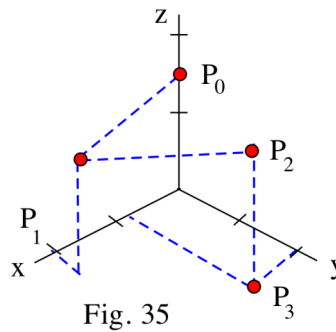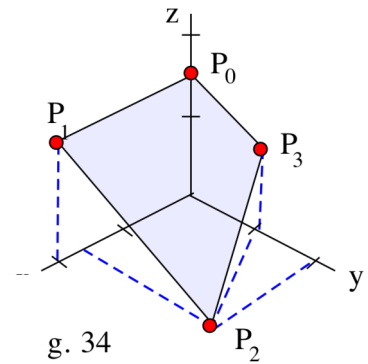
In problems 45 – 48, graphs of vector–valued functions are given. At the points labeled A, B, and C on each curve determine whether each variable function x(t), y(t), and z(t) is increasing (I) or decreasing (D) and fill in the table for each function.

45. $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is given in Fig. 37.

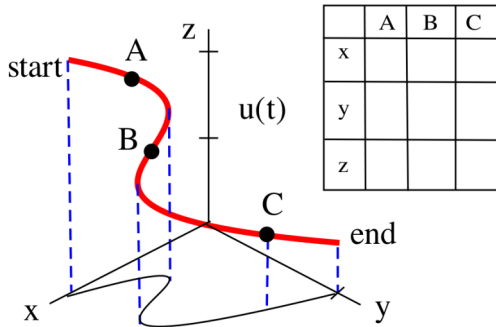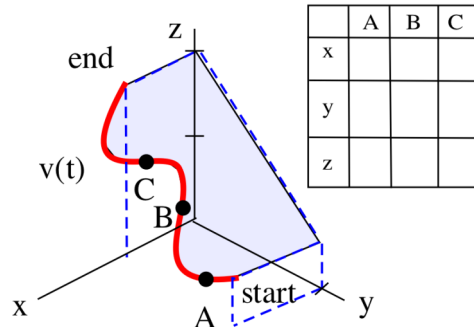46. $\mathbf{s}(t) = \langle x(t), y(t), z(t) \rangle$ is given in Fig. 38.

Fig. 37

	A	B	C
x			
y			
z			

Fig. 38

	A	B	C
x			
y			
z			
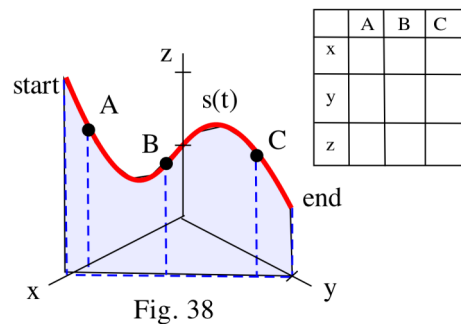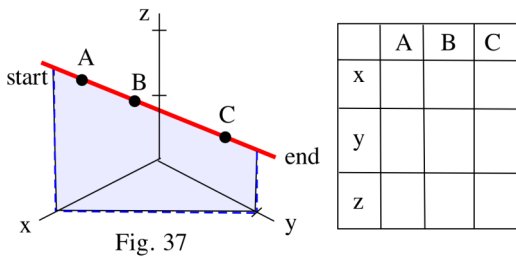
47. $\mathbf{u}(t) = \langle\, x(t), y(t), z(t)\,\rangle$ is given in Fig. 39.

	A	B	C
x			
y			
z			

Fig. 39

48. $\mathbf{v}(t) = \langle\, x(t), y(t), z(t)\,\rangle$ is given in Fig. 40.

	A	B	C
x			
y			
z			

Fig. 40

Practice Answers

Practice 1: The graphs of $\mathbf{r}(t) = \langle\, t, 4 - 2t, 2 - t\,\rangle$

and $\mathbf{s}(t) = \langle\, t, t^2, 1\,\rangle$ for $0 \le t \le 2$ are

shown in Fig. 41.

Practice 2: The graphs of x(t), y(t), and z(t) are

shown in Fig. 42a, 42b, and 42c. The graph of

$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = \langle\, x(t), y(t), z(t)\,\rangle$ is shown in
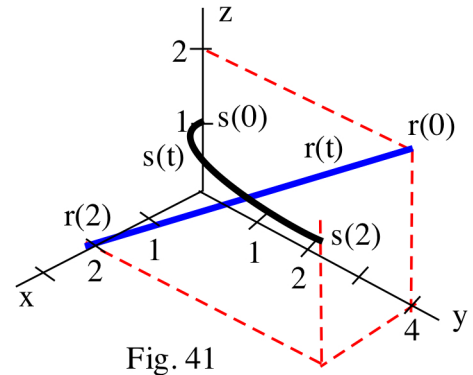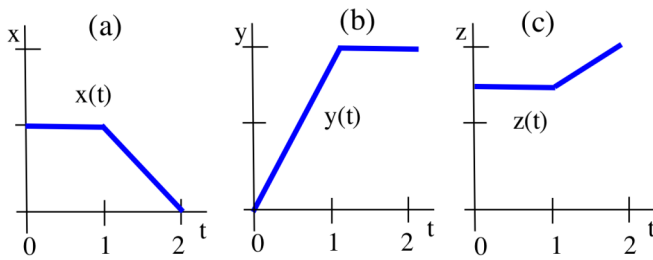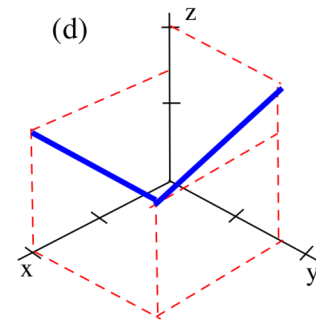
Fig. 42d .

Fig. 41

Fig. 42

Practice 3: For $\mathbf{r}(t) = \langle\, \cos(t)\,, \dfrac{\sin(t)}{t}\,, 3\,\rangle$, $\lim\limits_{t\varnothing\pi} \mathbf{r}(t) = \langle\, -1\,, 0\,, 3\,\rangle$, and $\lim\limits_{t\to\infty} \mathbf{r}(t)$ does not exist

because $\lim\limits_{t\to\infty} \cos(t)$ does not exist.

Practice 4: $\mathbf{r}(t) = \langle\, 1/t\,, e^t, \text{INT}(\,t\,)\,\rangle$ is continuous everywhere except where t is an integer since

INT(t) is not continuous where t is an integer. 1/t is not continuous when t = 0, but we

have already excluded t = 0 because 0 is an integer.

Practice 5:   The graphs of $\mathbf{r}(t) = \langle t-1, 0, 1+2t \rangle$ and $\mathbf{s}(t) = \langle 2, 4, 3t \rangle$ are

shown in Fig. 43 .

Practice 6:   The graph of $\mathbf{r}(t) = \langle t, \cos(t), \sin(t) \rangle$ is a circular helix around

the x–axis. This graph is shown in Fig. 44.

The graph of $\mathbf{s}(t) = \langle 2 \cdot \cos(t), t, 3 \cdot \sin(t) \rangle$ is an elliptical helix

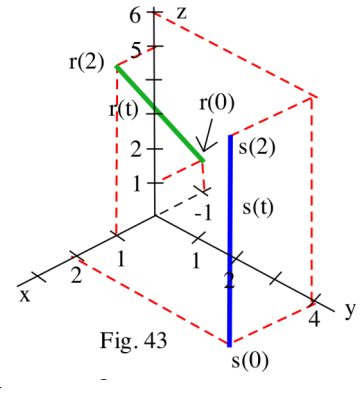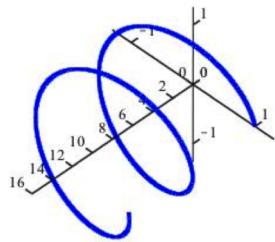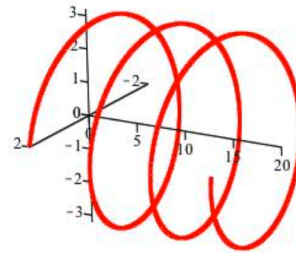around the y–axis. This graph is shown in Fig. 45.

Fig. 43

circular helix around the x-axis

Fig. 44

elliptical helix around the y-axis

Fig. 45

Appendix: MAPLE and graphs of vector–valued functions — spacecurve( )

The computer language MAPLE has a number of commands for creating graphs of 3–dimensional objects
including vector–valued functions. In order to access these commands, we first need to load the "plots" package:

with(plots); *(then press the <enter> key)*   This loads the "plots" package and lists the new commands
                                                available for us to use.

The command to create graphs of vector–valued functions is

   spacecurve([x(t), y(t), z(t)] , t = a..b, *options*);

where   x(t), y(t), and z(t) are formulas for the x, y, and z
coordinates,

a and b are the starting and stopping values for the variable t

and *options* includes commands for the type of axes, the color,

the thickness, and the number of points to be plotted.

Fig. M1

For example, the command

   spacecurve([t, sqrt(t), sin(t)], t=0..2*Pi, axes=NORMAL, numpoints=200, color=red, thickness=3);
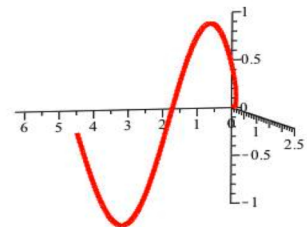
creates the graph in Fig. M1.

By positioning the cursor on the graph, a "hand" appears and, while holding

down the mouse button, we can rotate the graph by slowing moving the mouse

and then releasing the button. Fig. M2 shows

another view of the graph.

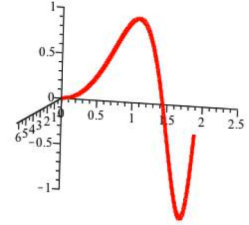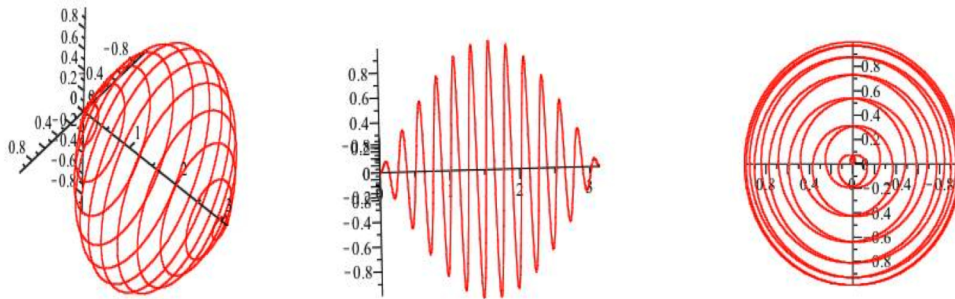Fig M2

Fig. M3 shows three views created using the command

spacecurve([sin(t)*cos(25*t), t, sin(t)*sin(25*t)], t=0..Pi, axes=NORMAL, numpoints=300);

Fig. M3: three views of [sin(t)*cos(25t), t , sin(t)*sin(25t)]