11.7 Vector Reflections

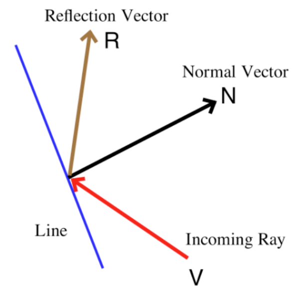

Figure 1

Vectors were originally used in physics and engineering (forces, work, ...), but a very common use now is in computer graphics, and a single frame in an animated movie such as "Cars" may require working with hundreds of millions of vectors. Realistic lighting and surfaces require calculating what happens when a ray is reflected off of a mirror or shiny surface. We will start examining that situation in 2D (Fig. 1), a ray reflected off of a line or curve, and later extend it to 3D, a ray reflecting off of a surface. The Reflection Theorem illustrates a modern use of the projection vector.

Reflection Theorem:

If an incident vector ray **V** is reflected off of a line L that has normal vector **N**,

then the reflection vector is $\mathbf{R} = \mathbf{V} - 2 \cdot \mathrm{Proj}_{\mathbf{N}}\mathbf{V}$. (Fig. 1)

Proof: The basis of the proof is very visual and geometric (Fig. 2). Put (i) $\mathbf{A} = \mathrm{Proj}_{\mathbf{N}}\mathbf{V}$, (ii) $\mathbf{B} = -2 \cdot \mathrm{Proj}_{\mathbf{N}}\mathbf{V}$, and (iii) $\mathbf{C} = \mathbf{V} - 2 \cdot \mathrm{Proj}_{\mathbf{N}}\mathbf{V}$. Then we can use geometry to justify that the vector **C** in (iii) is the reflection vector we want.

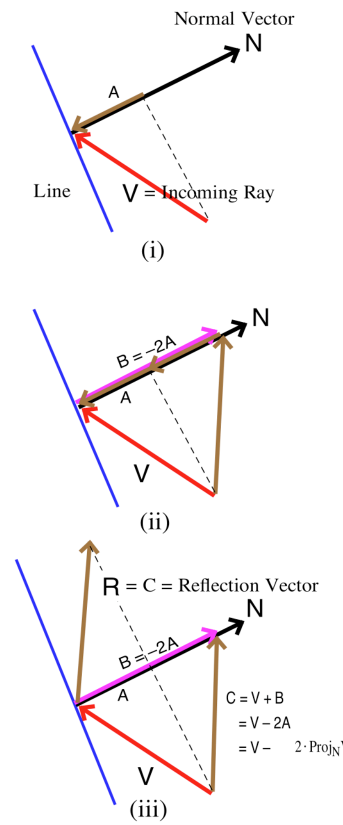

Fig. 2

Example 1: Find the reflection vector R when an incoming vector $\mathbf{V} = \langle -3, 1 \rangle$ is reflected by the line 3x + 2y = 12 (Fig. 3). Then give parametric equations for the reflected line when the reflection occurs at the point (2, 3).

Solution: The line has normal vector $\mathbf{N} = \langle 3, 2 \rangle$ so

$$\mathrm{Proj}_{\mathbf{N}}\mathbf{V} = \frac{\mathbf{V} \bullet \mathbf{N}}{\mathbf{N} \bullet \mathbf{N}}\mathbf{N} = \frac{-7}{13}\langle 3, 2 \rangle = \left\langle \frac{-21}{13}, \frac{-14}{13} \right\rangle$$

and the reflection vector is

$$\mathbf{R} = \mathbf{V} - 2 \cdot \mathrm{Proj}_{\mathbf{N}}\mathbf{V} = \langle -3, 1 \rangle - 2\left\langle \frac{-21}{13}, \frac{-14}{13} \right\rangle$$

$$= \left\langle -3 + \frac{42}{13}, 1 + \frac{28}{13} \right\rangle = \left\langle \frac{3}{13}, \frac{41}{13} \right\rangle$$

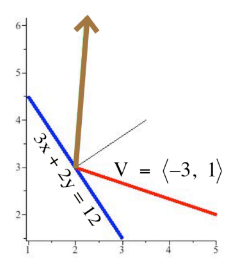

Fig. 3

One representation of the reflected line is $x(t) = 2 + \frac{3}{13}t, \ y(t) = 3 + \frac{41}{13}t$.

Since this operation is done millions of times for each frame, it is important that the operation be simple and quick.

Practice 1: Find the reflection vector R when an incoming vector $\mathbf{V} = \langle -1, -2 \rangle$ is reflected by the line –x + 2y = 2 (Fig. 4). Give parametric equations for the reflected line when the reflection occurs at the point (1.5, 1.75).

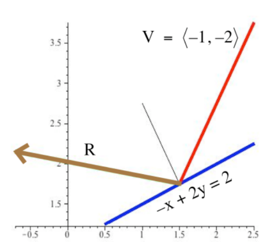

Fig. 4

We can also reflect off a curve at a point – just reflect off the tangent line to the curve at the point.

Example 2: Find the reflection vector R when an incoming vector $\mathbf{V} = \langle 3, -1 \rangle$ is reflected by the ellipse (2cos(t), sin(t)) at the point where t = 1.2 on the ellipse (Fig. 5).

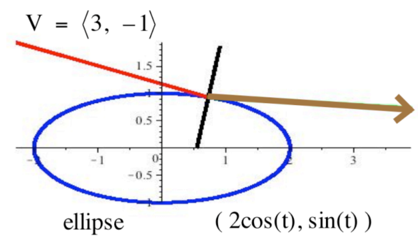

Fig. 5

Solution: $x'(t) = -2\sin(t)$ and $y'(t) = \cos(t)$ so a tangent vector to the ellipse when t = 1.2 is $\mathbf{T} = \langle -2\sin(1.2), \cos(1.2) \rangle \approx \langle -1.864, 0.362 \rangle$ and then a normal vector is $\mathbf{N} \approx \langle 0.362, 1.864 \rangle$. Finally,

$$\text{Proj}_{\mathbf{N}}\mathbf{V} = \frac{\mathbf{V} \bullet \mathbf{N}}{\mathbf{N} \bullet \mathbf{N}}\mathbf{N} \approx \frac{-0.778}{3.606}\langle 0.362, 1.864 \rangle = \langle -0.078, -0.402 \rangle$$

so $\mathbf{R} = \mathbf{V} - 2 \cdot \text{Proj}_{\mathbf{N}}\mathbf{V} = \langle 3, -1 \rangle - 2\langle -0.078, -0.402 \rangle = \langle 3.156, \text{-}0.197 \rangle$

Practice 2: Find the reflection vector **R** when an incoming vector $\mathbf{V} = \langle -1, 2 \rangle$ is reflected by the parabola (t, t^2) at the point where t = 1.2 on the parabola.

These ideas and computations extend very nicely to reflection vectors in 3D, and those reflections use the same formula: $\mathbf{R} = \mathbf{V} - 2 \cdot \text{Proj}_{\mathbf{N}}\mathbf{V}$. In the case of the reflection of a vector by a plane, we already know how to quickly find a normal vector to the plane so the reflection calculation is straightforward. To reflect off some other surface we will need a normal vector, and we will see how to find such a normal vector in Section 13.4.

Practice 3: Find the reflection vector **R** when an incoming vector $\mathbf{V} = \langle 2, -3, -1 \rangle$ is reflected by the plane 2x – 3y + z = 10 at the point where (5, 1, 3) on the plane.

Reflection Problems

For Problems 1 – 6, (a) determine the reflection vector **R** when an incoming vector **V** is reflected by the given line, and (b) determine parametric equations for the reflected line when the reflection occurs at the given point.

1. The incoming vector $\mathbf{V} = \langle 2, -1 \rangle$ is reflected by the line 3x + y = 6 at the point (1, 3).
2. The incoming vector $\mathbf{V} = \langle -1, 1 \rangle$ is reflected by the line 3x + y = 6 at the point (1, 3).
3. The incoming vector $\mathbf{V} = \langle 2, 3 \rangle$ is reflected by the line 5x – 2y = 7 at the point (3, 4).
4. The incoming vector $\mathbf{V} = \langle 0, -2 \rangle$ is reflected by the line 5x – 2y = 7 at the point (3, 4).

Problems 5 and 6 can be easily done without the Reflection Theorem, but, of course, the theorem works.

5. The incoming vector $\mathbf{V} = \langle -3, 2 \rangle$ is reflected by the y-axis at the point (0, 3).
6. The incoming vector $\mathbf{V} = \langle 3, -1 \rangle$ is reflected by the x-axis at the point (2, 0).

For Problems 7 – 12, (a) determine the reflection vector **R** when an incoming vector **V** is reflected by the given curve, and (b) determine parametric equations for the reflected line when the reflection occurs at the given point.

7. The incoming vector $\mathbf{V} = \langle -3, 1 \rangle$ is reflected by the ellipse (2cos(t), sin(t)) at the point where t = 0.9 on the ellipse.

8. The incoming vector $\mathbf{V} = \langle 2, 1 \rangle$ is reflected by the ellipse (2cos(t), sin(t)) at the point where t = 0.9 on the ellipse.

9. The incoming vector $\mathbf{V} = \langle 2, 1 \rangle$ is reflected by the curve $\left(t^2, t^3 \right)$ at the point where t = 2.

10. The incoming vector $\mathbf{V} = \langle -1, 1 \rangle$ is reflected by the curve $\left(t^2, t^3 \right)$ at the point where t = 2.

11. The incoming vector $\mathbf{V} = \langle -1, 1 \rangle$ is reflected by the curve $y = x^2$ at the point (2, 4).

12. The incoming vector $\mathbf{V} = \langle 2, 1 \rangle$ is reflected by the curve $y = x^2$ at the point (2, 4).

For Problems 13 – 16, (a) determine the reflection vector R when an incoming vector V is reflected by the given plane, and (b) determine parametric equations for the reflected line when the reflection occurs at the given point.

13. The incoming vector $V = \langle 2, 6, 3 \rangle$ is reflected by the plane x + 2y + 3z = 13 at the point (2, 4, 1).

14. The incoming vector $V = \langle 4, 1, 3 \rangle$ is reflected by the plane 3x – 2y + 4z = 5 at the point (1, 3, 2).

15. The incoming vector $V = \langle 3, 2, 1 \rangle$ is reflected by the plane x = 0 at the point (0, 4, 2).

16. The incoming vector $V = \langle 2, -3, -1 \rangle$ is reflected by the plane z = 0 at the point (3, 4, 0).

Practice Answers

Practice 1: The line has normal vector $\mathbf{N} = \langle -1, 2 \rangle$ so $\text{Proj}_{\mathbf{N}}\mathbf{V} = \frac{\mathbf{V} \bullet \mathbf{N}}{\mathbf{N} \bullet \mathbf{N}}\mathbf{N} \approx \frac{-3}{5}\langle -1, 2 \rangle = \left\langle \frac{3}{5}, \frac{-6}{5} \right\rangle$

and the reflection vector is $\mathbf{R} = \mathbf{V} - 2 \cdot \text{Proj}_{\mathbf{N}}\mathbf{V} = \langle -1, -2 \rangle - 2\left\langle \frac{3}{5}, \frac{-6}{5} \right\rangle = \left\langle \frac{-11}{5}, \frac{2}{5} \right\rangle$.

The reflected line is $x(t) = 1.5 - \frac{11}{5}t, \ y(t) = 1.75 + \frac{2}{5}t$.

Practice 2: $x'(t) = 1$ and $y'(t) = 2t$ so a tangent vector to the parabola when $t = 1.2$ is $\mathbf{T} = \langle 1, 2.4 \rangle$ and then a normal vector is $\mathbf{N} \approx \langle -2.4, 1 \rangle$. Finally, $\text{Proj}_{\mathbf{N}}\mathbf{V} = \dfrac{\mathbf{V} \bullet \mathbf{N}}{\mathbf{N} \bullet \mathbf{N}}\mathbf{N} \approx \dfrac{4.4}{6.76}\langle -2.4, 1 \rangle = \langle -1.56, 0.65 \rangle$

so $\mathbf{R} = \mathbf{V} - 2 \cdot \text{Proj}_{\mathbf{N}}\mathbf{V} = \langle -1, 2 \rangle - 2\langle -1.56, 0.65 \rangle = \langle 2.12, 0.7 \rangle$

$y(t) = 0.783 - 2.099t$.

Related web sites:

Ray tracing in Wikipedia – a nice overview

http://en.wikipedia.org/wiki/Ray_tracing_(graphics)

Just look at the pictures in this one (about movie "Cars" from Pixar)

http://graphics.pixar.com/library/RayTracingCars/paper.pdf

"Rendering this image used **111 million diffuse rays, 37 million specular rays, and 26 million shadow rays**. The rays cause **1.2 billion ray-triangle intersection tests**. With multiresolution geometry caching, the render time is 106 minutes."