11.5 CROSS PRODUCT

This section is the final one about the arithmetic of vectors, and it introduces a second type of vector–vector multiplication called the cross product. The material in this section and the previous sections is the foundation for the next several chapters on the calculus of vector–valued functions and functions of several variables, and all of the vector arithmetic is used extensively.

The dot product of two vectors results in a scalar, a number related to the magnitudes of the two original vectors and to the angle between them, and the dot product is defined for two vectors in 2–dimensional, 3–dimension, and higher dimensional spaces. The cross product of a vector and a vector differs from the dot product in several significant ways: the cross product is only defined for two vectors in 3–dimensional space, and the cross product of two vectors is a vector. At first, the definition of the cross product given below may seem strange, but the resulting vector has some very useful properties as well as some unusual ones. The torque wrench described next illustrates some of the properties we get with the cross product.

Torque wrench: As you pull down on the torque wrench in Fig. 1, a force is applied to the bolt that twists it into the wall. This "twisting" force is the result of two vectors, the length and a direction of the wrench $\mathbf{A}$ and the magnitude and direction of the pulling force $\mathbf{B}$. To model this "twisting into the wall" result of $\mathbf{A}$ and $\mathbf{B}$, we want a vector $\mathbf{C}$ that points into the wall and depends on the magnitudes of $\mathbf{A}$ and $\mathbf{B}$ as well as on the angle between the wrench and the direction of the pull.

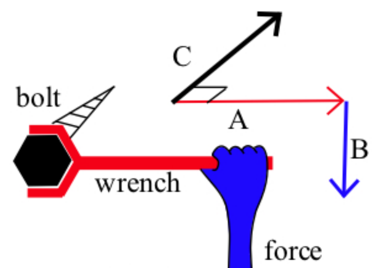


Fig. 1

If the pull is downward (Fig. 1), we want $\mathbf{C}$ to point into the page.

If the pull is upward (Fig. 2), we want $\mathbf{C}$ to point out of the page.

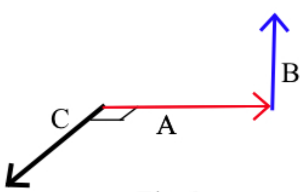


Fig. 2

If the angle between $\mathbf{A}$ and $\mathbf{B}$ is close to $\pm 90^{\circ}$ (Fig. 3), we want the magnitude of $\mathbf{C}$ to be large.

If the angle between $\mathbf{A}$ and $\mathbf{B}$ is small (Fig. 4), we want the magnitude of $\mathbf{C}$ to be small.

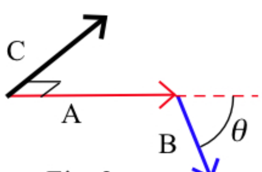


Fig. 3

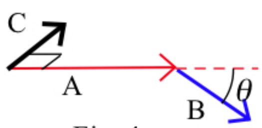


Fig. 4

There is a way to combine the vectors $\mathbf{A}$ and $\mathbf{B}$ to produce a vector $\mathbf{C}$ with the properties that model the torque wrench. This vector $\mathbf{C}$, called the cross product of $\mathbf{A}$ and $\mathbf{B}$, also turns out to be very useful when we discuss planes through given points and tangent planes to surfaces. It is not at all obvious from the definition given below for the cross product that the cross product has a relationship to torque wrenches, planes, or anything else of interest or use to us, but it does and we will investigate those applied and geometric properties. Try not to be repelled by the unusual definition — it leads to some lovely results.

Definition of the **Cross Product (Vector Product):**

For $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$, the **cross product** of **A** and **B** is

$$\mathbf{A} \times \mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}\text{, a vector.}$$

The symbols "$\mathbf{A} \times \mathbf{B}$" are read "**A** cross **B**."
Many people find it difficult to remember a complicated formula like the definition of the cross product. Fortunately, there is an easy way to do so, but it requires a digression into the calculation of determinants.

Determinants

Determinants appear in a number of areas of mathematics, and you may have already seen them as part of Cramer's Rule for solving systems of equations. Here we only need them as a device for making it easier to remember and calculate and use the cross product.

Definition of the 2x2 ("two by two") Determinant: The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Some people prefer to remember the visual pattern in Fig. 5.

Example 1: Evaluate the determinants: $\begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix}$, $\begin{vmatrix} x & y \\ -2 & 3 \end{vmatrix}$, and $\begin{vmatrix} \mathbf{i} & \mathbf{j} \\ 0 & 4 \end{vmatrix}$.

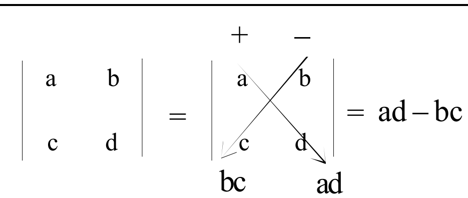


Fig. 5: Pattern for a 2x2 determinant

Solution: $\begin{vmatrix} 1 & 4 \\ 3 & 5 \end{vmatrix} = (1)(5) - (4)(3) = 5 - 12 = -7.$

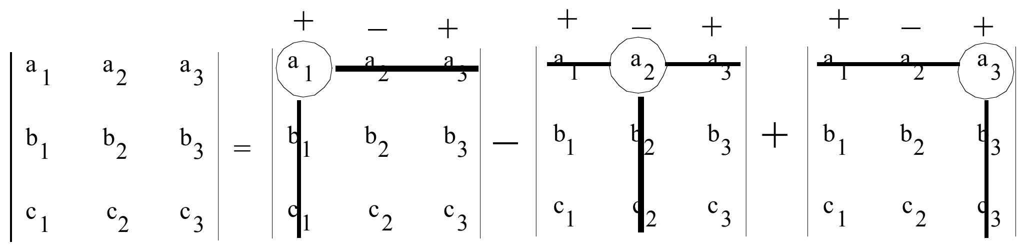
$\begin{vmatrix} x & y \\ -2 & 3 \end{vmatrix} = (x)(3) - (y)(-2) = 3x + 2y$.

$\begin{vmatrix} \mathbf{i} & \mathbf{j} \\ 0 & 4 \end{vmatrix} = (\mathbf{i})(4) - (\mathbf{j})(0) = 4\mathbf{i}$.

Practice 1: Evaluate the determinants: $\begin{vmatrix} -3 & 4 \\ 5 & 6 \end{vmatrix}$, $\begin{vmatrix} x & y \\ 0 & -3 \end{vmatrix}$, and $\begin{vmatrix} \mathbf{i} & \mathbf{j} \\ -4 & 3 \end{vmatrix}$.

A 3x3 determinant can be defined in terms of several 2x2 determinants.

Definition of the 3x3 Determinant: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$.



$$= +a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Fig. 6: Pattern for a 3x3 determinant

Many people prefer to remember the visual pattern in Fig. 6.

In the 3x3 definition, the first 2x2 determinant, $\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}$, is the part of the original 3x3 table after the first row and the first column have been removed, the row and column containing a_1. The second 2x2 determinant $\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}$ is what remains of the original table after the first row and the second column have been removed, the row and column containing a_2. The third 2x2 determinant $\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$ is what remains of the original table after the first row and the third column have been removed, the row and column containing a_3. (Note: The leading signs attached to the three terms alternate: **+ – +**.)

Example 2: Evaluate the determinants $\begin{vmatrix} 2 & 3 & 5 \\ 0 & -4 & 1 \\ -3 & 4 & 0 \end{vmatrix}$ and $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 1 \\ 0 & -2 & 5 \end{vmatrix}$.

Solution: $\begin{vmatrix} 2 & 3 & 5 \\ 0 & -4 & 1 \\ -3 & 4 & 0 \end{vmatrix} = (2)\begin{vmatrix} -4 & 1 \\ 4 & 0 \end{vmatrix} - (3)\begin{vmatrix} 0 & 1 \\ -3 & 0 \end{vmatrix} + (5)\begin{vmatrix} 0 & -4 \\ -3 & 4 \end{vmatrix} = (2)(-4) - (3)(3) + (5)(-12) = -77.$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 1 \\ 0 & -2 & 5 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 4 & 1 \\ -2 & 5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 1 \\ 0 & 5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 4 \\ 0 & -2 \end{vmatrix} = +22\mathbf{i} - 15\mathbf{j} + (-6)\mathbf{k}.$$

Practice 2: Evaluate the determinants $\begin{vmatrix} 3 & 5 & 0 \\ 1 & 4 & -1 \\ -2 & 0 & 6 \end{vmatrix}$ and $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & 0 & 5 \end{vmatrix}$.

The original definition of the cross product can now be rewritten using the determinant notation.

Determinant Form of the Cross Product Definition:

For $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.

The second determinant in Example 2 represents the cross product $\mathbf{A} \times \mathbf{B}$ for $\mathbf{A} = \langle 3, 4, 1 \rangle$ and $\mathbf{B} = \langle 0, -2, 5 \rangle$, and the second determinant in Practice 2 is the cross product $\mathbf{A} \times \mathbf{B}$ for $\mathbf{A} = \langle 2, -1, 3 \rangle$ and $\mathbf{B} = \langle 4, 0, 5 \rangle$.

The cross products of various pairs of the basis vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are relatively easy to evaluate, and they begin to illustrate some of the properties of cross products.

Example 3: Use the determinant form of the definition of the cross product to evaluate (a) $\mathbf{i} \times \mathbf{j}$, (b) $\mathbf{j} \times \mathbf{i}$, and (c) $\mathbf{i} \times \mathbf{i}$.

Solution:

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k} = \mathbf{k}.$$

$$\mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (-1)\mathbf{k} = -\mathbf{k}.$$

$$\mathbf{i} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$$

You should note that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ is perpendicular to the vectors $\mathbf{i}$ and $\mathbf{j}$ and the xy–plane. Similarly, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ is also perpendicular to $\mathbf{i}$ and $\mathbf{j}$ and the xy–plane.

Practice 3: Use the determinant form of the definition of the cross product to evaluate (a) $\mathbf{j} \times \mathbf{k}$, (b) $\mathbf{k} \times \mathbf{j}$, and (c) $\mathbf{j} \times \mathbf{j}$.

The cross products of pairs of basis vectors follow a simple pattern given below and in Fig. 7.

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$	$\mathbf{i} \times \mathbf{i} = \mathbf{0}$
$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$	$\mathbf{j} \times \mathbf{j} = \mathbf{0}$

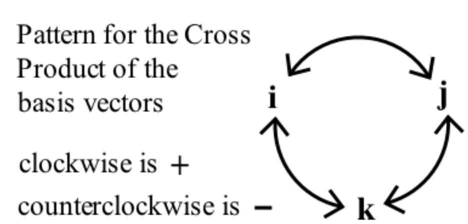


Fig. 7

$\mathbf{k} \times \mathbf{i} = \mathbf{j}$ $\qquad$ $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ $\qquad$ $\mathbf{k} \times \mathbf{k} = \mathbf{0}$

Example 4: Evaluate $\mathbf{A} \times \mathbf{B}$ for $A = \langle 2, -3, 0 \rangle$ and $B = \langle 3, 1, -4 \rangle$.

Solution:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 3 & 1 & -4 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -3 & 0 \\ 1 & -4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 0 \\ 3 & -4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix} = 12\mathbf{i} + 8\mathbf{j} + 11\mathbf{k}\ .$$

Practice 4: Evaluate $\mathbf{A} \times \mathbf{B}$ for $A = \langle 3, 0, -5 \rangle$ and $B = \langle -2, 4, 1 \rangle$.

Properties of the Cross Product:

(a) $\mathbf{0} \times \mathbf{A} = \mathbf{A} \times \mathbf{0} = \mathbf{0}$

(b) $\mathbf{A} \times \mathbf{A} = \mathbf{0}$

(c) $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$

(d) $k(\mathbf{A} \times \mathbf{B}) = k\mathbf{A} \times \mathbf{B} = \mathbf{A} \times k\mathbf{B}$

(e) $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$

The proofs of all of these properties are straightforward applications of the definition of the cross product as is illustrated below for part (c). Proofs of (a) and (b) are given in the Appendix after the Practice Answers, and the proofs of (d) and (e) are left as exercises.

Proof of (c): If $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{B} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ a_1 & a_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}$$

$$= \mathbf{i}\,(a_3b_2 - a_2b_3) - \mathbf{j}\,(a_3b_1 - a_1b_3) + \mathbf{k}\,(a_2b_1 - a_1b_2)\ .$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \mathbf{i}\,(a_2b_3 - a_3b_2) - \mathbf{j}\,(a_1b_3 - a_3b_1) + \mathbf{k}\,(a_1b_2 - a_2b_1) = -\mathbf{B} \times \mathbf{A}\ .$$

A vector has both a direction and a magnitude, and the direction and magnitude of the vector $\mathbf{A} \times \mathbf{B}$ each give us useful information: the direction of $\mathbf{A} \times \mathbf{B}$ is perpendicular to $\mathbf{A}$ and $\mathbf{B}$, and the magnitude of $\mathbf{A} \times \mathbf{B}$ is the area of the parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$. These are the two properties of the cross product that get used most often in the later chapters, and they enable us to visualize $\mathbf{A} \times \mathbf{B}$.

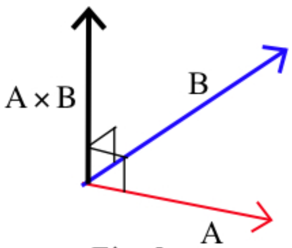


Fig. 8

The **direction** property of **A** x **B**:

A•(**A** x **B**) = 0 and **B**•(**A** x **B**) = 0

so **A** x **B** is perpendicular to **A** and to **B**. (Fig. 8)

Proof: $\mathbf{A}\bullet(\mathbf{A} \times \mathbf{B}) = \langle a_1, a_2, a_3 \rangle \bullet \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

$= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1 = 0$.

The proof that **B**•(**A** x **B**) = 0 is similar .

The **magnitude** property of **A** x **B**:

If **A** and **B** are nonzero vectors with angle θ between them $(0 \le \theta \le \pi)$,

then $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \, |\sin(\theta)|$

= **area of the parallelogram** formed by the vectors **A** and **B**. (Fig. 9)

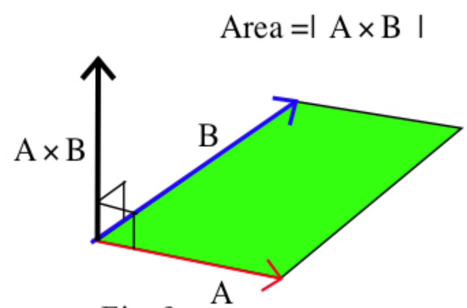


Fig. 9

The proof that $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \, |\sin(\theta)|$ is algebraically complicated and is given in the Appendix after the Practice Answers. These properties of the direction and magnitude are sometimes used to define the cross product, and then the algebraic definition is derived from them.

Corollary to the magnitude property of **A** x **B**:

The **area of the triangle** formed by the vectors **A** and **B** is $\frac{1}{2} |\mathbf{A} \times \mathbf{B}|$.

Visualizing the cross product: The cross product **A** x **B** is perpendicular to both **A** and **B** and satisfies a "right hand rule" so we can visualize the direction of **A** x **B** using, of course, your right hand.

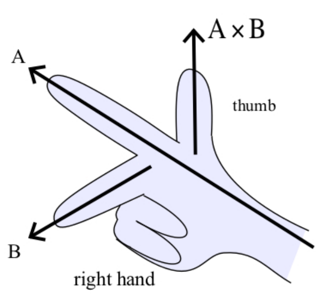


Fig. 10

If **A** and **B** are the indicated fingers (Fig. 10 and Fig. 11)) then your extended right thumb points in the direction of **A** x **B**.

The relative length of **A** x **B** can be estimated from the magnitude property of **A** x **B** : | **A** x **B** | = parallelogram area.

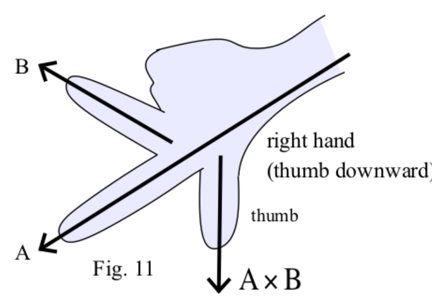


Fig. 11

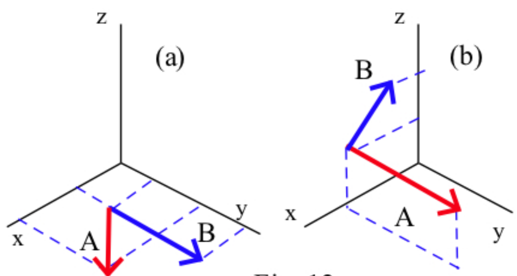


Fig. 12

Example 5: Fig. 12 shows two pairs of vectors **A** and **B**. For each pair sketch the direction of **A** x **B** . For which pair is | **A** x **B** | larger?

Solution: See Fig. 13. | **A** x **B** | is larger for (b) since the area of their parallelogram is larger in (b).

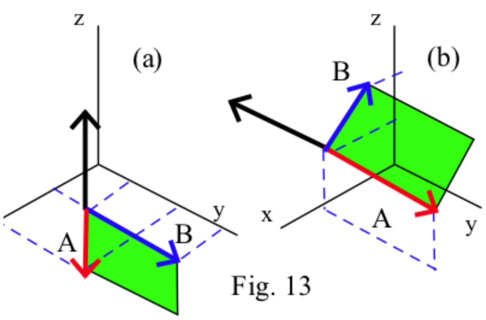


Fig. 13

Practice 5: Fig. 14 shows two pairs of vectors **A** and **B**. For each pair sketch the direction of **A** x **B** . For which pair is | **A** x **B** | larger?

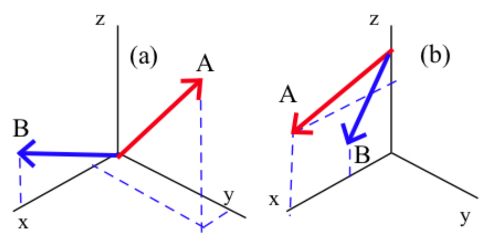


Fig. 14

Torque Wrench Revisited

Now we can use the ideas and properties of the cross product to analyze the original torque wrench problem.

> **Definition**: The **torque vector** produced by a lever arm vector **A** and a force vector **B** is **A** x **B** .

The direction of the torque vector tells us whether the wrench is driving the bolt into the wall or pulling it out of the wall. The magnitude of the torque vector describes the strength of the tendency of the wrench to drive the bolt in or pull it out.

Example 6: Fig. 15 shows two force vectors acting at the end of a 10 inch wrench. Which vector produces the larger torque?

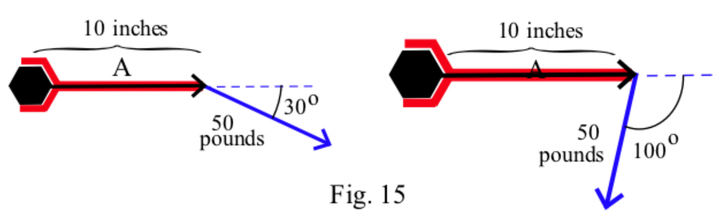


Fig. 15

Solution: |**B**| = 50 pounds with $\theta = 30°$ so

| **A** x **B** | = |**A**||**B**| |sin(θ)|

= (10 inches)(50 pounds) |sin(30°)| ≈ 250 pound–inches of force.

|**C**| = 50 pounds with $\theta = 100°$ so

| **A** x **C** | = |**A**||**C**| |sin(θ)| = (10 inches)(50 pounds)| sin(100°) | ≈ 492.4 pound–inches of force.

Vector **C** produces the larger torque: the smaller force, used intelligently, produced the larger result.

Practice 6: In Fig. 16 which force vector produces the larger torque?

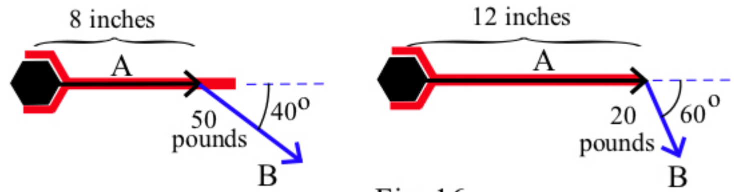


Fig. 16

The Triple Scalar Product A•(B x C)

The combination of the dot and cross products, **A**•(**B** x **C**) , for the three vectors **A**, **B**, and **C** in 3–dimensional space is called the triple scalar product because the result of these operations is a scalar:**A**•(**B** x **C**) = **A**•(vector) = scalar. And the magnitude of this scalar has a nice geometric meaning.

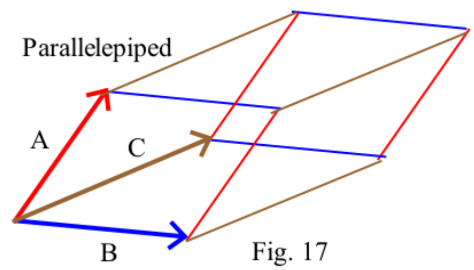


Fig. 17

Geometric meaning of | **A**•(**B** x **C**) |

For the vectors **A**, **B**, and **C** in 3–dimensional space (Fig.17)

| **A**•(**B** x **C**) | = volume of the parallelepiped (box) with sides **A**, **B**, and **C** .

Proof: From the definition of the dot product,

A•(**B** x **C**) = | **B** x **C** | |**A**| cos(θ) so

| **A**•(**B** x **C**) | = | **B** x **C** | |**A**| |cos(θ)| .

Volume = (area of the base)(height),

and the area of the base of the box is | **B** x **C** | .

Since **B** x **C** is perpendicular to the base, the height h is the projection of **A** onto **B** x **C** (Fig. 18): h = |**A**| | cos(θ) |.

Then Volume = | **B** x **C**| |**A**| | cos(θ) | which we showed was equal to | **A**•(**B** x **C**) | .

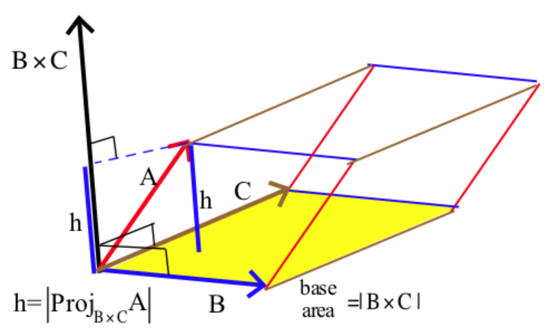


Fig. 18

Problem 46 asks you to show that the triple scalar product can be calculated as a 3x3 determinant.

Beyond Three Dimensions

The objects we examined in previous sections (points, distances, vectors, dot products, angles between vectors, projections) all had rather nice extensions to more than three dimensions. The cross product is different: the cross product **A** x **B** we have defined requires that **A** and **B** be 3–dimensional vectors, and there is no easy extension to vectors in more than three dimensions that preserves the properties of the cross product.

PROBLEMS

In problems 1 – 12, evaluate the determinants.

1. $\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix}$ 2. $\begin{vmatrix} 4 & -1 \\ 3 & 1 \end{vmatrix}$ 3. $\begin{vmatrix} x & 5 \\ y & 2 \end{vmatrix}$ 4. $\begin{vmatrix} 5 & a \\ b & 3 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ 6. $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ 7. $\begin{vmatrix} 1 & 3 & 2 \\ 0 & 5 & 2 \\ 1 & 1 & 0 \end{vmatrix}$ 8. $\begin{vmatrix} 2 & 3 & 0 \\ 1 & -3 & 2 \\ -1 & 0 & 4 \end{vmatrix}$

9. $\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$ 10. $\begin{vmatrix} a & b & c \\ 0 & 3 & 5 \\ 2 & 1 & 3 \end{vmatrix}$ 11. $\begin{vmatrix} 2 & 3 & 5 \\ 0 & -4 & 1 \\ -3 & 4 & 0 \end{vmatrix}$ 12. $\begin{vmatrix} x & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1-x \end{vmatrix}$

In problems 13 – 18, vectors **A** and **B** are given. Calculate (a) **A** x **B** , (b) (**A** x **B**)•**A**, (c) (**A** x **B**)•**B** , and (d) | **A** x **B** | .

13. $\mathbf{A} = \langle 3, 4, 5 \rangle$, $\mathbf{B} = \langle -1, 2, 0 \rangle$ 14. $\mathbf{A} = \langle -2, 2, 2 \rangle$, $\mathbf{B} = \langle 3, 1, 2 \rangle$

15. $\mathbf{A} = \langle 1, -3, 2 \rangle$, $\mathbf{B} = \langle -2, 6, 4 \rangle$ 16. $\mathbf{A} = \langle 6, 8, -2 \rangle$, $\mathbf{B} = \langle 3, 4, -1 \rangle$

17. $\mathbf{A} = 3\mathbf{i} - 1\mathbf{j} + 4\mathbf{k}$, $\mathbf{B} = 1\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ 18. $\mathbf{A} = 4\mathbf{i} - 1\mathbf{j} + 0\mathbf{k}$, $\mathbf{B} = 3\mathbf{i} - 2\mathbf{j} + 0\mathbf{k}$

In problems 19 – 22, state whether the result of the given calculation is a vector, a scalar, or is not defined.

19. **A**•(**B** x **C**)   20. **A** x (**B**•**C**)   21. (**A**•**B**) x (**A**•**C**)   22. **A**(**B** x **C**)

23. Prove property (d) of the Properties of the Cross Product: k(**A** x **B**) = k**A** x **B** = **A** x k**B** .

24. Prove property (e) of the Properties of the Cross Product: **A** x (**B** + **C**) = (**A** x **B**) + (**A** x **C**) .

25. Explain geometrically why **A** x **A** = **0** .

26. If |**A**| and |**B**| are fixed, what angle(s) between **A** and **B** maximizes | **A** x **B** | ? Why?

In problems 27 – 30, vectors **A** and **B** are given graphically. Sketch **A** x **B** .

27. See Fig. 19.   28. See Fig. 20.   29. See Fig. 21.   30. See Fig. 22.

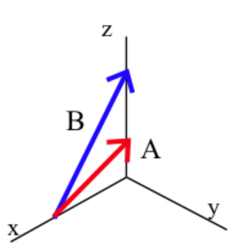


Fig. 19

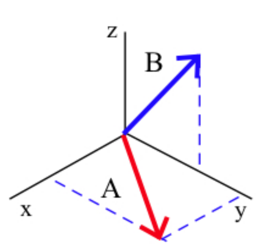


Fig. 20

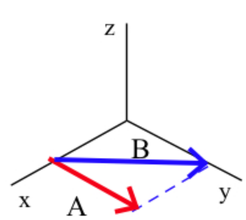


Fig. 21

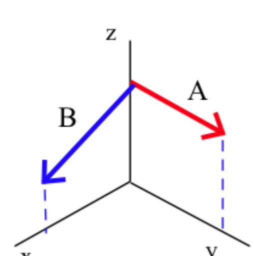


Fig. 22

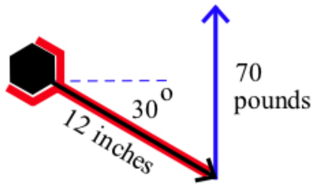


Fig. 23

31. (a) Calculate the torque produced by the wrench and force shown in Fig. 23.

(b) Calculate the torque produced by the wrench and force shown in Fig. 24.

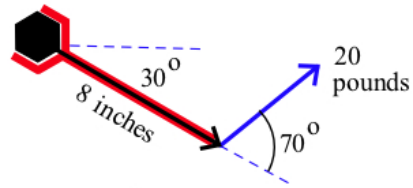


Fig. 24

32. When a tire nut is "frozen" (stuck), a pipe is sometimes put over the handle of the tire wrench (Fig. 25). Using the vocabulary and ideas of vectors, explain why this is effective.

Fig. 25

33. Does the torque on wrench **A** produced by **B** plus the torque produced by **C** equal the torque produced by **B** + **C** ? Why or why not?

Areas

34. Sketch the parallelogram with sides $\mathbf{A} = \langle 5, 1, 0 \rangle$ and $\mathbf{B} = \langle 2, 4, 0 \rangle$ and find its area. Sketch and find the area of the triangle with sides **A** and **B** .

35. Sketch the parallelogram with sides $\mathbf{A} = \langle 1, 2, 0 \rangle$ and $\mathbf{B} = \langle 0, 4, 2 \rangle$ and find its area. Sketch and find the area of the triangle with sides **A** and **B** .

36. Sketch the triangle with vertices P = (4, 0, 1), Q =(1, 3, 1), and R = (2, 0, 5) and find its area.

37. Sketch the triangle with vertices P = (1, 4, –2), Q =(3, 5, 1), and R = (5, 2, 2) and find its area.

38. Sketch the triangle with vertices P = (a, 0, 0), Q =(0, b, 0), and R = (0, 0, c) and find its area.

Triple Scalar Products

39. Sketch a parallelepiped with edges $\mathbf{A} = \langle 2, 1, 0 \rangle$, $\mathbf{B} = \langle -1, 4, 1 \rangle$, $\mathbf{C} = \langle 1, 1, 2 \rangle$ and find its volume.

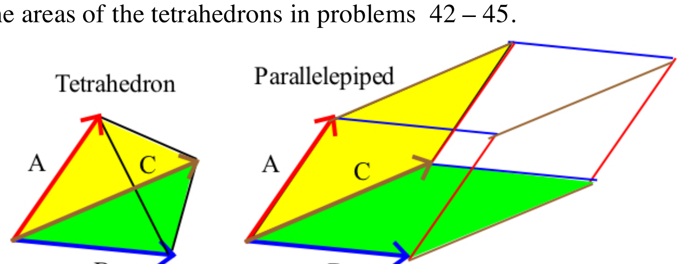
40. Sketch a parallelepiped with edges $\mathbf{A} = \langle 2, 0, 3 \rangle$, $\mathbf{B} = \langle 0, 4, 5 \rangle$, $\mathbf{C} = \langle 4, 3, 0 \rangle$ and find its volume.

41. Sketch a parallelepiped with edges $\mathbf{A} = \langle a, 0, 0 \rangle$, $\mathbf{B} = \langle 0, b, 0 \rangle$, $\mathbf{C} = \langle 0, 0, c \rangle$ and find its volume.

Use the result that "the volume of the tetrahedron with edges **A**, **B**, **C** (Fig. 26) is 1/6 the volume of the parallelepiped with the same edges" to find the areas of the tetrahedrons in problems 42 – 45.

42. Sketch the tetrahedron with vertices P = (0,0,0), Q =(3,1,0), R = (0,4,0), and S = (0,0,3) and find its volume.

43. Sketch the tetrahedron with vertices P = (1,0,2), Q =(3,1,2), R = (0,4,3), and S = (0,1,4) and find its volume.



Tetrahedron volume = $\frac{1}{6}$(Parallelepiped volume)
Fig. 26

44. Sketch the tetrahedron with vertices $P = (0,0,0)$, $Q = (2,0,0)$, $R = (0,4,0)$, and $S = (0,0,6)$ and find its volume.

45. Sketch the tetrahedron with vertices $P = (0,0,0)$, $Q = (a,0,0)$, $R = (0,b,0)$, and $S = (0,0,c)$ and find its volume.

46. Show that if $\mathbf{A} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{B} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{C} = \langle c_1, c_2, c_3 \rangle$,

then $$\mathbf{A} \bullet (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} .$$

Right Tetrahedrons

47. The four points $(0,0,0)$, $(2,0,0)$, $(0,4,0)$, and $(0,0,4)$ form a tetrahedron (Fig. 27) with four triangular faces. Find the areas A_{xy}, A_{xz}, A_{yz}, and A_{xyz} of the four triangular faces.

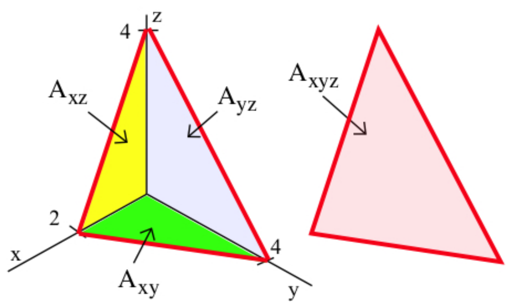


Fig. 27

48. The four points $(0,0,0)$, $(2,0,0)$, $(0,4,0)$, and $(0,0,6)$ form a tetrahedron (Fig. 28) with four triangular faces. Find the areas A_{xy}, A_{xz}, A_{yz}, and A_{xyz} of the four triangular faces.

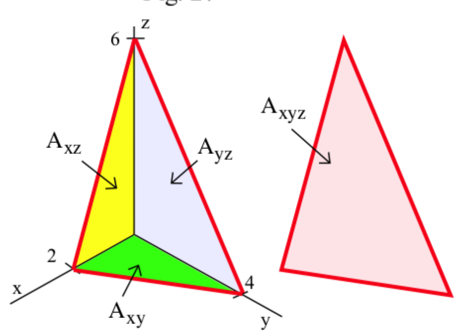


Fig. 28

49. Verify that the answers to problems 47 and 48 satisfy the relationship $(A_{xy})^2 + (A_{xz})^2 + (A_{yz})^2 = (A_{xyz})^2$.

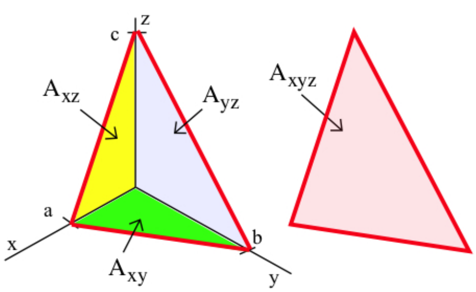


Fig. 29

50. For the right tetrahedron with vertices $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, determine the areas of the four triangular faces (Fig. 29) and prove the Pythagorean type result for areas of triangles in a right tetrahedron:

$$(A_{xy})^2 + (A_{xz})^2 + (A_{yz})^2 = (A_{xyz})^2 .$$

51. The Pythagorean pattern
$a^2 + b^2 = c^2$
can be thought of as relating a line segment C in two dimensions and its "shadows" a and b on the coordinate axes. Show that this "shadow" interpretation also holds for the area of a triangle in three dimensions and the areas of its "shadows" on the three coordinate planes (Fig. 30):

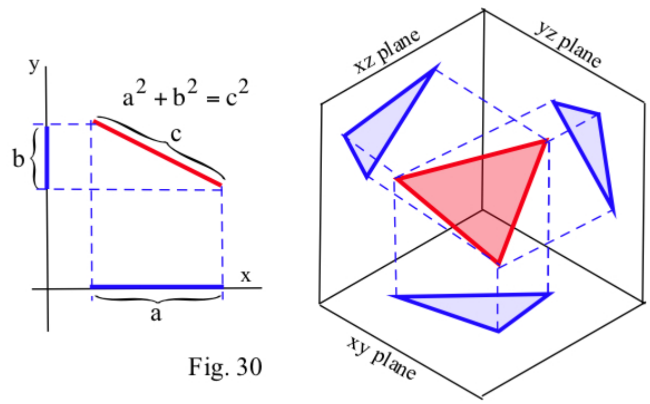


Fig. 30

$$(\text{area of dark triangle})^2 = (\text{xy shadow area})^2 + (\text{xz shadow area})^2 + (\text{yz shadow area})^2 .$$

Areas of Regions in the Plane

Among its several uses, the cross product also leads to a simple, easily programmed algorithm for finding the area of a "simple" (no edges cross) polygon in the plane, and this algorithm is used to approximate the areas of other regions as well.

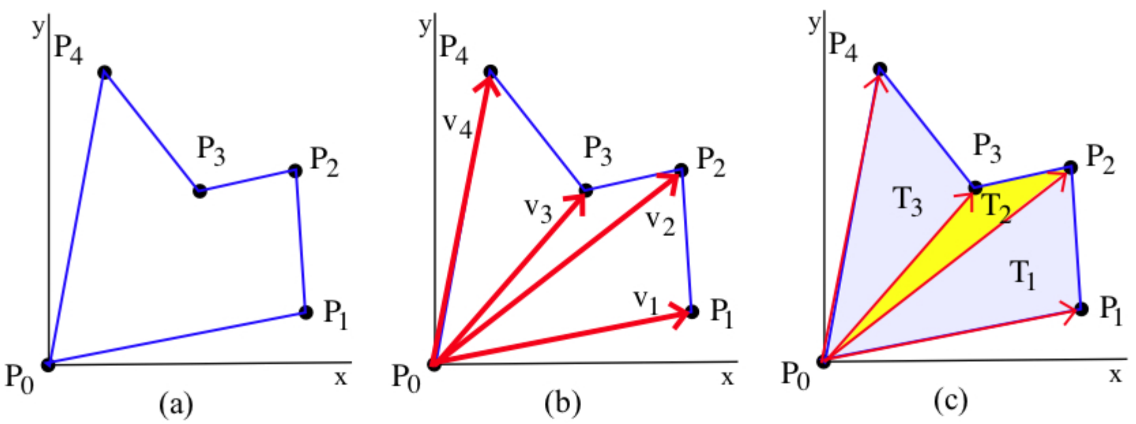


Fig. 31

Suppose $P_0 = (0, 0)$, $P_1 = (x_1, y_1)$, ... , $P_4 = (x_4, y_4)$ are 5 vertices of a simple polygon (Fig. 31a), with one vertex at the origin and then labeling the others as we travel counterclockwise around the polygon. Let $\mathbf{V}_1$ be the vector from P_0 to P_1, $\mathbf{V}_2$ from P_0 to P_2, ... (Fig. 31b). Then the area of the polygon in Fig. 31c is the sum of the 3 triangular areas T_1, T_2, and T_3 and each triangular area can be found using a cross product: $T_1 = \frac{1}{2} \mid \mathbf{V}_1 \times \mathbf{V}_2 \mid = \frac{1}{2}(x_1y_2 - x_2y_1)$,

$T_2 = \frac{1}{2} \mid \mathbf{V}_2 \times \mathbf{V}_3 \mid = \frac{1}{2}(x_2y_3 - x_3y_2)$, and $T_3 = \frac{1}{2} \mid \mathbf{V}_3 \times \mathbf{V}_4 \mid = \frac{1}{2}(x_3y_4 - x_4y_4)$.

Finally, the total area is the sum

$$\text{Area} = T_1 + T_2 + T_3$$

$$= \frac{1}{2} \{ (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_4 - x_4y_3) \}$$

$$= \frac{1}{2} \sum_{k=1}^{n-1} (x_k y_{k+1} - x_{k+1} y_k) \quad \text{with } n = 4 .$$

The last summation formula works for polygons with at least three vertices. In fact, this algorithm is used by computers to report the area of a region traced by a cursor or stylus: the computer reads the (x,y) location of the cursor several times per second and uses the data and this algorithm to calculate the area of the region (as approximated by a many–sided polygon).

52. Use the given pattern to find the area of the rectangle with vertices (0,0), (2,0), (2,3), and (0,3). Does the pattern give the area of the rectangle?

53. Use the given pattern to find the area of the pentagon with vertices (0,0), (4,1), (5,3), (4,4), (2,4), and (1,3).

54. How can we modify the algorithm to handle the situation in which none of the vertices are at the origin? Show that your modification works for the rectangle with vertices (1,3), (3,3), (3,6), and (1,6).

Note: The cross product satisfies a "right hand rule" so if we go counterclockwise from **U** to **V** (Fig. 32a) then **U** x **V** is positive, and if we go clockwise from **U** to **V** (Fig. 32b) then **U** x **V** is negative.

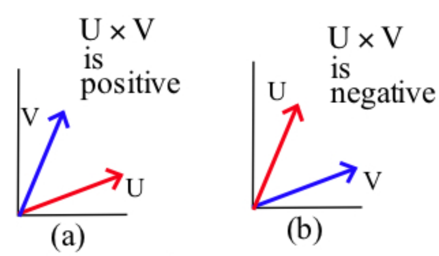


Fig. 32

55. In Fig. 33a, $\frac{1}{2}$ $\mathbf{V}_1 \times \mathbf{V}_2$ gives the area of T_1 as a positive number; $\frac{1}{2}$ $\mathbf{V}_2 \times \mathbf{V}_3$ gives the area of T_2 as a negative number; and $\frac{1}{2}$ $\mathbf{V}_3 \times \mathbf{V}_4$ gives the area of T_3 as a positive number.

Explain geometrically how these positive and negative numbers "fit together" to give the correct area for the region in Fig. 33b.

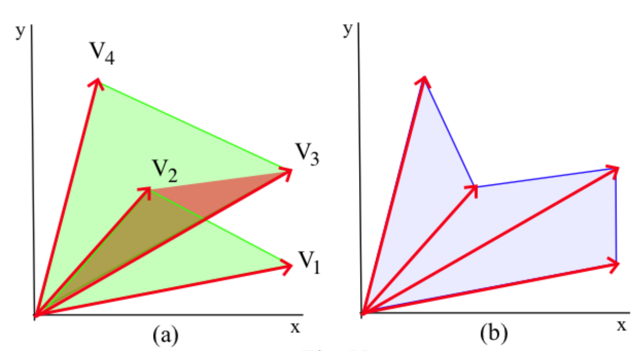


Fig. 33

Practice Answers

Practice 1: $\begin{vmatrix} -3 & 4 \\ 5 & 6 \end{vmatrix} = (-3)(6) - (4)(5) = -38.$ $\begin{vmatrix} x & y \\ 0 & -3 \end{vmatrix} = (x)(-3) - (y)(0) = -3x$.

$\begin{vmatrix} \mathbf{i} & \mathbf{j} \\ -4 & 3 \end{vmatrix} = (\mathbf{i})(3) - (\mathbf{j})(-4) = 3\mathbf{i} + 4\mathbf{j}$.

Practice 2: $\begin{vmatrix} 3 & 5 & 0 \\ 1 & 4 & -1 \\ -2 & 0 & 6 \end{vmatrix} = (3)\begin{vmatrix} 4 & -1 \\ 0 & 6 \end{vmatrix} - (5)\begin{vmatrix} 1 & -1 \\ -2 & 6 \end{vmatrix} + (0)\begin{vmatrix} 1 & 4 \\ -2 & 0 \end{vmatrix} = (3)(24) - (5)(4) + (0)(8) = 52.$

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & 0 & 5 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 3 \\ 0 & 5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = -5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

Practice 3: $\mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 1\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{i}$.

$\mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$. $\mathbf{j} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{0}$.

Practice 4: $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -5 \\ -2 & 4 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & -5 \\ 4 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -5 \\ -2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 0 \\ -2 & 4 \end{vmatrix} = 20\mathbf{i} + 7\mathbf{j} + 12\mathbf{k}$.

Practice 5: See Fig. 34. The pair in (a) produces the larger torque.

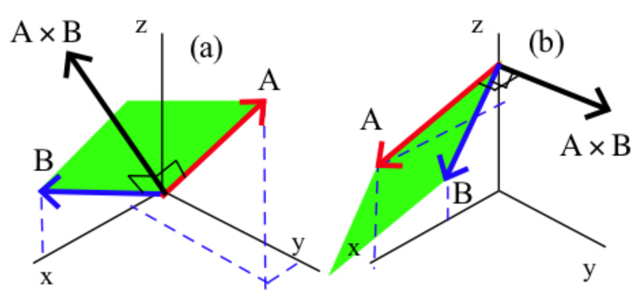


Fig. 34

Practice 6: For **B**,

| torque | = (8 inches)(50 pounds) sin(40°) ≈ 257.1 inch–pounds.

For **C**, | torque | = (12 inches)(20 pounds) sin(60°) ≈ 207.8 inch–pounds.

Force **B** produces the larger torque: sometimes strength is enough.

Appendix: Some Proofs

Proof of (a): **0** x **A** = **A** x **0** = **0.** If **A** = $\langle a_1, a_2, a_3 \rangle$, then

$$\mathbf{0} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} .$$

The proof that **A** x **0** = **0** is similar.

Proof of (b): **A** x **A** = **0.** If **A** = $\langle a_1, a_2, a_3 \rangle$, then

$$\mathbf{A} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ a_1 & a_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0} .$$

Proof that $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\,|\sin(\theta)|$:

$$\mathbf{A} \times \mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \quad \text{so}$$

$$\begin{aligned}
|\mathbf{A} \times \mathbf{B}|^2 &= (\mathbf{A} \times \mathbf{B})\bullet(\mathbf{A} \times \mathbf{B}) \\
&= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2b_3a_3b_2 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_3b_1a_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1b_2a_2b_1 + a_2^2b_1^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \quad \text{(expand \& check)} \\
&= |\mathbf{A}|^2|\mathbf{B}|^2 - (\mathbf{A}\bullet\mathbf{B})^2 \\
&= |\mathbf{A}|^2|\mathbf{B}|^2 - (|\mathbf{A}||\mathbf{B}|\cos(\theta))^2 \qquad \text{since } \mathbf{A}\bullet\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos(\theta) \\
&= |\mathbf{A}|^2|\mathbf{B}|^2 - |\mathbf{A}|^2|\mathbf{B}|^2\cos^2(\theta) \\
&= |\mathbf{A}|^2|\mathbf{B}|^2 \{ 1 - \cos^2(\theta) \} \\
&= |\mathbf{A}|^2|\mathbf{B}|^2 \sin^2(\theta) .
\end{aligned}$$

Then, taking the square root of each side of $|\mathbf{A} \times \mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2 \sin^2(\theta)$, we have

$$|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|\,|\sin(\theta)| .$$