Chapter 10: Odd Answers

Section 10.0

1. (a) 32, 64 (b) 2^5 (c) 2^n

3. (a) –1, +1 (b) $-1 = (-1)^5$ (c) $(-1)^n$

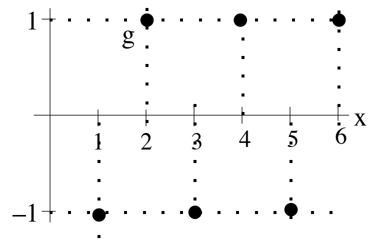
5. (a) 120, 720 (b) 5! or 5•24 (c) n! or $n \cdot a_{n-1}$

7. 1, 3/2, 11/6, 25/12, 137/60, 147/60

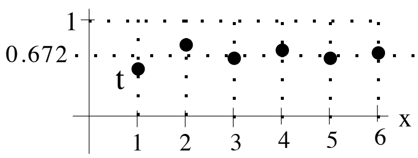
9. 1, 1/2, 3/4, 5/8, 11/16, 21/32

11. 1, 0, 1, 0, 1, 0

13. (a) g(5) = –1, g(6) = +1 (b) see the figure for g(x).

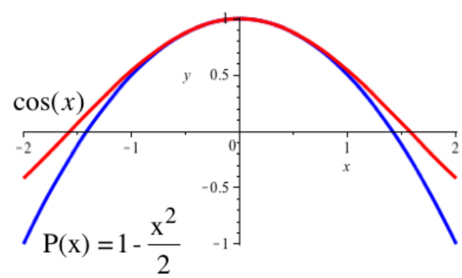


15. (a) t(5) = 1 – 1/2 + 1/4 – 1/8 + 1/16 – 1/32 = 21/32, t(6) = 43/64 . The graph of t(x) is shown.



17. (a) $P(x) = 1 - \frac{x^2}{2}$ (b) Graphs

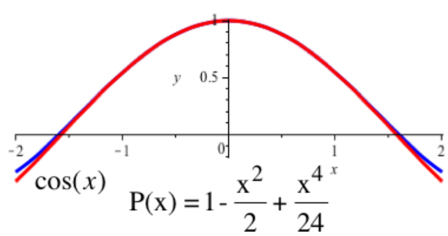
x	P(x)	cos(x)	| P(x) – cos(x) |
0	1.0	1.0	0
0.1	0.995	0.99500	0
0.2	0.98	0.98006	0.00006
0.3	0.955	0.95533	0.00033
1.0	0.5	0.54030	0.04030
2.0	–1.0	–0.41615	0.58385



(c) $P(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$

Graphs

x	P(x)	cos(x)	| P(x) – cos(x) |
0	1.0	1.0	0
0.1	0.99500	0.99500	0
0.2	0.98006	0.98006	0
0.3	0.95533	0.95533	0
1.0	0.54167	0.54030	0.00137
2.0	–0.33333	–0.41615	0.08282



19. P(x) = Ax + B. 5 = P(0) = A•0 + B = B so B = 5. 3 = P '(0) = A. P(x) = 3x + 5.

21. P(x) = Ax + B. 4 = P(0) = A•0 + B = B so B = 4. –1 = P '(0) = A. P(x) = –1x + 4.

23. P(x) = 0x + 4 = 4.

25. P(x) = Ax + B. P(0) = B. P '(0) = A.

27. $P(x) = Ax^2 + Bx + C$. –2 = P(0) = A•0 + B•0 + C = C. 7 = P '(0) = 2A•0 + B = B. 6 = P "(0) = 2A so A = 6/2 = 3. $P(x) = 3x^2 + 7x - 2$.

29. $P(x) = Ax^2 + Bx + C$. 8 = P(0) = A•0 + B•0 + C = C. 5 = P '(0) = 2A•0 + B = B. 10 = P "(0) = 2A so A = 10/2 = 5. $P(x) = 5x^2 + 5x + 8$.

31. $P(x) = Ax^2 + Bx + C$. $-3 = P(0) = A\cdot 0 + B\cdot 0 + C = C$. $-2 = P'(0) = 2A\cdot 0 + B = B$.
$4 = P''(0) = 2A$ so $A = 4/2 = 2$. $P(x) = 2x^2 - 2x - 3$.

33. $P(x) = Ax^3 + Bx^2 + Cx + D$. $5 = P(0) = A\cdot 0 + B\cdot 0 + C\cdot 0 + D = D$. $3 = P'(0) = 3A\cdot 0 + 2B\cdot 0 + C = C$.
$4 = 6A0 + 2B = 2B$ so $B = 4/2 = 2$. $6 = P'''(0) = 6A$ so $A = 6/6 = 1$.
$P(x) = 1x^3 + 2x^2 + 3x + 5$.

35. $P(x) = Ax^3 + Bx^2 + Cx + D$. $4 = P(0) = A\cdot 0 + B\cdot 0 + C\cdot 0 + D = D$. $-1 = P'(0) = 3A\cdot 0 + 2B\cdot 0 + C = C$.
$-2 = 6A0 + 2B = 2B$ so $B = -2/2 = -1$. $-12 = P'''(0) = 6A$ so $A = -12/6 = -2$.
$P(x) = -2x^3 - 1x^2 - 1x + 4$.

37. $P(x) = Ax^3 + Bx^2 + Cx + D$. $4 = P(0) = A\cdot 0 + B\cdot 0 + C\cdot 0 + D = D$. $0 = P'(0) = 3A\cdot 0 + 2B\cdot 0 + C = C$.
$-4 = 6A0 + 2B = 2B$ so $B = -4/2 = -2$. $36 = P'''(0) = 6A$ so $A = 36/6 = 6$.
$P(x) = 6x^3 - 2x^2 + 0x + 4 = 6x^3 - 2x^2 + 4$.

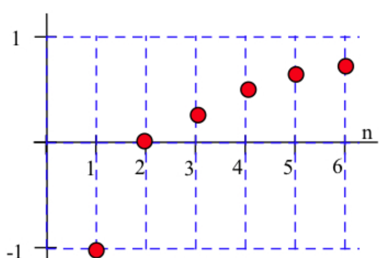
39. $A = P'''(0)/6$, $B = P''(0)/2$, $C = P'(0)$, and $D = P(0)$.

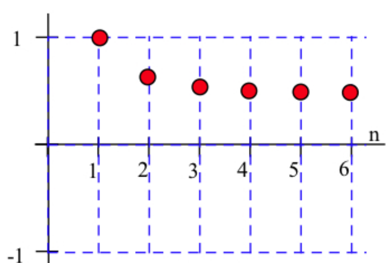
Section 10.1

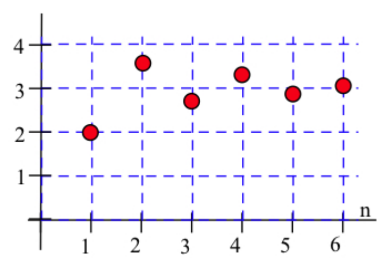
1. $\frac{1}{n^2}$ 3. $\frac{n-1}{n} = 1 - \frac{1}{n}$ 5. $\frac{n}{2^n}$

7. $\{ -1, 0, 1/3, 1/2, 3/5, 2/3, \ldots \}$ Graph is shown. 9. $\{ 1, 2/3, 3/5, 4/7, 5/9, 6/11, \ldots \}$ Graph is shown.

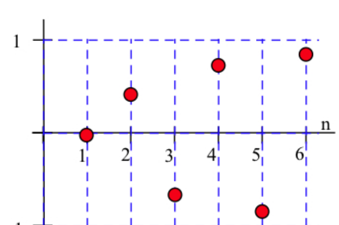
11. $\{ 2, 3\frac{1}{2}, 2\frac{2}{3}, 3\frac{1}{4}, 2\frac{4}{5}, 3\frac{1}{6}, \ldots \}$ Graph is shown. 13. $\{ 0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, \ldots \}$ Graph is shown.

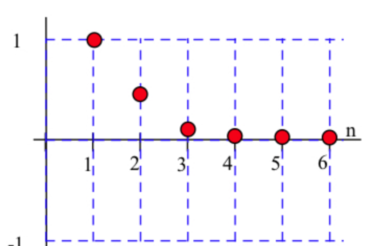

Prob. 7

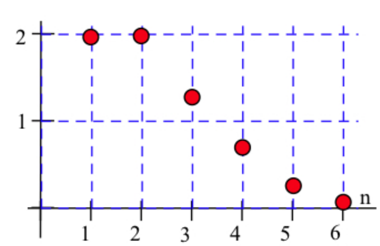

Prob. 9


Prob. 11

15. $\{ 1, \frac{1}{2}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \frac{1}{6!}, \ldots \} = \{ 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \ldots \}$ The graph is shown below.


Prob. 13


Prob. 15


Prob. 17

17. $\{ \frac{2^1}{1!}, \frac{2^2}{2!}, \frac{2^3}{3!}, \frac{2^4}{4!}, \frac{2^5}{5!}, \frac{2^6}{6!}, \ldots \}$ The graph is shown.

19. $a_1 = 2, a_2 = -2, a_3 = 2, a_4 = -2, a_5 = 2, a_6 = -2, a_7 = 2, a_8 = -2, a_9 = 2, a_{10} = -2$

21. $\{ \sin(2\pi/3), \sin(4\pi/3), \sin(6\pi/3), \sin(8\pi/3), \sin(10\pi/3), \sin(12\pi/3), \sin(14\pi/3), \sin(16\pi/3),$
$\sin(18\pi/3), \sin(20\pi/3), \ldots \}$

23. $c_1 = 1, c_2 = 3, c_3 = 6, c_4 = 10, c_5 = 15, c_6 = 21, c_7 = 28, c_8 = 36, c_9 = 45, c_{10} = 55$

25. $\{ a_n \}$ appears to converge. $\{ b_n \}$ does not appear to converge.

27. $\{ e_n \}$ does not appear to converge. $\{ f_n \}$ appears to converge.

29. $\left\{ 1 - \frac{2}{n} \right\}$ converges to 1.

31. $\left\{ \frac{n^2}{n+1} \right\}$ grows arbitrarily large and diverges.

33. $\left\{ \frac{n}{2n-1} \right\}$ converges to $\frac{1}{2}$

35. $\left\{ \ln(3 + \frac{7}{n} \right\}$ converges to $\ln(3) \approx 1.099$.

37. $\left\{ 4 + (-1)^n \right\}$ alternates in value between 3 and 5 and does not appraoch a single number.
The sequence diverges.

39. $\left\{ \frac{1}{n!} \right\}$ converges to 0.

41. $\left\{ (1 - \frac{1}{n})^n \right\}$ converges to $e^{-1} = \frac{1}{e}$. (See Section 3.7, Example 7.)

43. $\left\{ \frac{(n+2)(n-5)}{n^2} \right\} = \left\{ \frac{n^2 - 3n - 10}{n^2} \right\} = \left\{ 1 - \frac{3}{n} - \frac{10}{n^2} \right\}$ converges to 1.

45. Take $N = \sqrt{\frac{3}{\varepsilon}}$. If $n > N = \sqrt{\frac{3}{\varepsilon}}$ then $n^2 > \frac{3}{\varepsilon}$ and $\varepsilon > \frac{3}{n^2} = \left| \frac{3}{n^2} - 0 \right|$.

47. Take $N = \frac{1}{\varepsilon}$. If $n > N = \frac{1}{\varepsilon}$ then $\varepsilon > \frac{1}{n} = | (3 - \frac{1}{n}) - 3 | = | \frac{3n-1}{n} - 3 |$.

49. $\left\{ \frac{1}{n^{th} \text{ prime}} \right\}$ is a subsequence of $\left\{ \frac{1}{n^{th} \text{ integer}} \right\} = \left\{ \frac{1}{n} \right\}$ which converges to 0, so we can conclude that $\left\{ \frac{1}{n^{th} \text{ prime}} \right\}$ converges to 0.

51. $\left\{ (-2)^n (\frac{1}{3})^n \right\} = \left\{ (-1)^n (\frac{2}{3})^n \right\}$.

If n is even, $\left\{ (-1)^n (\frac{2}{3})^n \right\} = \left\{ (\frac{2}{3})^n \right\}$ which converges to 0. If n is odd, $\left\{ (-1)^n (\frac{2}{3})^n \right\} = \left\{ -(\frac{2}{3})^n \right\}$ which also converges to 0. Since "n even" and "n odd" account for all of the positive integers, we can conclude that $\left\{ (-2)^n (\frac{1}{3})^n \right\}$ converges to 0.

53. $\left\{ (1 + \frac{5}{n^2})^{(n^2)} \right\}$ is a subsequence of $\left\{ (1 + \frac{5}{n})^n \right\}$ which converges to e^5 (Section 3.7, Example 7)

so $\left\{ (1 + \frac{5}{n^2})^{(n^2)} \right\}$ also converges to e^5 .

55. $a_n = 7 - \frac{2}{n}$ so $a_{n+1} = 7 - \frac{2}{n+1}$. $a_{n+1} - a_n = (7 - \frac{2}{n+1}) - (7 - \frac{2}{n}) = \frac{2}{n} - \frac{2}{n+1} = \frac{2}{n(n+1)} > 0$ for all $n \geq 1$.

Therefore, $a_{n+1} > a_n$ = and $\{ a_n \}$ is monotonically increasing.

57. $a_n = 2^n$ so $a_{n+1} = 2^{n+1}$. $a_{n+1} - a_n = 2^{n+1} - 2^n = 2^n 2 - 2^n = 2^n(2 - 1) = 2^n > 0$ for all $n \geq 1$.

Therefore, $a_{n+1} > a_n$ = and $\{ a_n \}$ is monotonically increasing.

59. $a_n = \frac{n + 1}{n!}$ so $a_{n+1} = \frac{(n+1) + 1}{(n+1)!} = \frac{n + 2}{(n+1)!}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n + 2}{(n+1)!}}{\frac{n + 1}{n!}} = \frac{n + 2}{n + 1} \frac{n!}{(n+1)!} = \frac{n+2}{n+1} \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1)} = \frac{n + 2}{n + 1} \frac{1}{n + 1} < 1 \text{ for all } n \geq 1$$

so $a_{n+1} < a_n$ for all $n \geq 1$ and $\{ a_n \}$ is monotonically decreasing.

61. $a_n = (\frac{5}{4})^n$ so $a_{n+1} = (\frac{5}{4})^{n+1}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(\frac{5}{4})^{n+1}}{(\frac{5}{4})^n} = \frac{5}{4} > 1$$ for all n so $a_{n+1} > a_n$ for all $n > 0$ and $\{ a_n \}$ is montonically increasing.

63. $a_n = \frac{n}{e^n}$ so $a_{n+1} = \frac{n+1}{e^{n+1}}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}} = \frac{n+1}{n} \frac{e^n}{e^{n+1}} = \frac{n+1}{n} \frac{1}{e} < 1$$ for $n > 1$ (reason: $e > 2$ so $n \cdot e > 2n > n + 1$ so $\frac{n+1}{n \cdot e} < 1$).

So $a_{n+1} < a_n$ for all $n > 0$ and $\{ a_n \}$ is monotonically decreasing.

65. Let $f(x) = 5 - \frac{3}{x}$. Then $f'(x) = \frac{3}{x^2} > 0$ for all x so f(x) is increasing. From that we can conclude that $a_n = f(n)$ is monotonically increasing.

67. Let $f(x) = \cos(\frac{1}{x})$. Then $f'(x) = -\sin(\frac{1}{x}) \cdot (\frac{-1}{x^2}) = \frac{1}{x^2} \cdot \sin(\frac{1}{x}) > 0$ for all $x \geq 1$. From that we can conclude that $a_n = f(n)$ is monotonically increasing.

69. This is similar to problem 59. The ratio method works nicely.

71. One method is to examine $a_{n+1} - a_n = \left(1 - \frac{1}{2^{n+1}}\right) - \left(1 - \frac{1}{2^n}\right) = \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^n} - \frac{1}{2 \cdot 2^n} = \frac{1}{2 \cdot 2^n} > 0$ for all n so $a_{n+1} > a_n$ for all n and $\{ a_n \}$ is monotonically increasing.

The ratio and derivative methods also work.

($\mathbf{D}\left(1 - \frac{1}{2^x}\right) = \mathbf{D}(1 - 2^{-x}) = 0 - 2^{-x} \cdot \ln(2) \cdot \mathbf{D}(-x) = 2^{-x}\ln(2) = \frac{\ln(2)}{2^x} > 0$ for $x > 0$.)

73. This is similar to problem 63. The ratio method works nicely.

75. N = 4: $a_1 = 4, a_2 = \frac{1}{2}\left(4 + \frac{4}{4}\right) = \frac{5}{2} = 2.5$, $a_3 = \frac{1}{2}\left(2.5 + \frac{4}{2.5}\right) = 2.05$, $a_n = \frac{1}{2}\left(2.05 + \frac{4}{2.05}\right) \approx 2.00061$.

N = 9: $a_1 = 9, a_2 = \frac{1}{2}\left(9 + \frac{9}{9}\right) = \frac{10}{2} = 5$, $a_3 = \frac{1}{2}\left(5 + \frac{9}{5}\right) = 3.2$, $a_n = \frac{1}{2}\left(3.2 + \frac{9}{3.2}\right) = 3.00625$.

N = 5: $a_1 = 5, a_2 = \frac{1}{2}\left(5 + \frac{5}{5}\right) = \frac{6}{2} = 3$. $a_3 = \frac{1}{2}\left(3 + \frac{5}{3}\right) \approx 2.333$, $a_n = \frac{1}{2}\left(2.333 + \frac{5}{2.333}\right) \approx 2.238$.

77. (a) $p = 0.02$, and we want to solve $0.01 = \frac{0.02}{0.02k + 1}$ for k. Then $0.02k + 1 = \frac{0.02}{0.01} = 2$ so $0.02k = 1$ and $k = \frac{1}{0.02} = 50$ generations.

(b) We want to solve $\frac{1}{2}p = \frac{p}{kp + 1}$ for k in terms of p. $kp + 1 = \frac{p}{0.5p} = 2$ so $kp = 1$ and $k = \frac{1}{p}$ generations.

79. (a) The first "few" grains can be anywhere on the x–axis.

(b) After a "lot of grains" have been placed, there will be a large pile of sand close to 3 on the x–axis.

81. $-1 \le \sin(n) \le 1$ for all integers n.

(a) The first few grains will be scattered between -1 and $+1$ on the x–axis.

(b) After a "lot of grains" have been placed, the sand will be scattered "uniformly" along the interval from -1 to $+1$. (See part (c).)

(c) This argument is rather sophisticated, but the result is interesting: no two grains ever end up on the same point.

We assume that two grains do end up on the same point, and then derive a contradiction. From this we conclude that our original assumption (two grains on one point) was false.

Assume that two grains do end up on the same point so $a_m = a_n$ for distinct integers m and n. Then $\sin(m) = \sin(n)$ so $0 = \sin(m) - \sin(n) = 2 \cdot \sin\left(\frac{m-n}{2}\right) \cdot \cos\left(\frac{m+n}{2}\right)$ and either $\sin\left(\frac{m-n}{2}\right) = 0$ or $\cos\left(\frac{m+n}{2}\right) = 0$. If $\sin\left(\frac{m-n}{2}\right) = 0$, then $\frac{m-n}{2} = \pi K$ for some integer K and $\pi = \frac{m-n}{2K}$ where m, n, and K are integers. Then π is a rational number, a contradiction of the fact that π is irrational.

If $\cos\left(\frac{m+n}{2}\right) = 0$, then $\frac{m+n}{2} = \frac{\pi}{2} + K\pi = \pi\left(\frac{1}{2} + K\right)$ for some integer K so $\pi = \frac{m+n}{1 + 2K}$, a rational number. This again contradicts the irrationality of π, so out original assumption (two grains on the same point) was false.

Section 10.2

1. $\sum_{k=1}^{\infty} \frac{1}{k}$

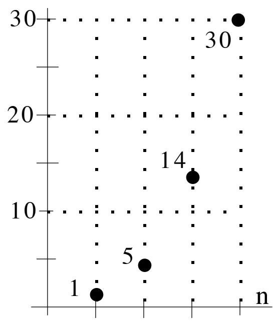
3. $\sum_{k=1}^{\infty} \frac{2}{3k}$

5. $\sum_{k=1}^{\infty} (-\frac{1}{2})^k$ or $\sum_{k=1}^{\infty} (-1)^k \cdot \frac{1}{2^k}$

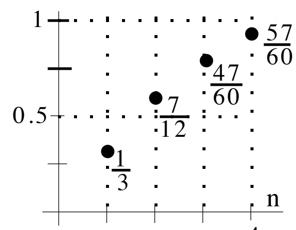
7. The graph is given.

9. The graph is given.

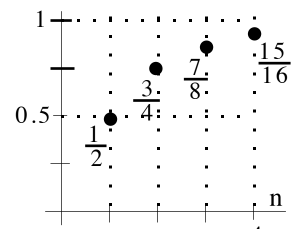
11. The graph is given.



Problem 7



Problem 9



Problem 11

13. $a_1 = 3$, $a_2 = -1$, $a_3 = 2$, $a_4 = 1$

15. $a_1 = 4$, $a_2 = 0.5$, $a_3 = -0.2$, $a_4 = 0.5$

17. $a_1 = 1$, $a_2 = 0.1$, $a_3 = 0.01$, $a_4 = 0.001$

19. $\sum_{k=1}^{\infty} \frac{8}{10^k}$

21. $\sum_{k=1}^{\infty} \frac{5}{10^k}$

23. $\sum_{k=1}^{\infty} \frac{a}{10^k}$

25. $\sum_{k=1}^{\infty} \frac{17}{100^k}$

27. $\sum_{k=1}^{\infty} \frac{7}{100^k}$

29. $\sum_{k=1}^{\infty} \frac{abc}{1000^k}$

31. $\sum_{k=0}^{\infty} 30 \cdot (0.8)^k$

33. 80, 64, 51.2, $100 \cdot (0.8)^n$

35. $(1/4)^n \to 0$ so the series may converge or may diverge. (Later we will see that it converges.)

37. $(4/3)^n \to \infty \neq 0$ so the series diverges.

39. $\frac{\sin(n)}{n} \to 0$ so the series may converge or may diverge.

41. $\cos(1/n) \to \cos(0) = 1 \neq 0$ so the series diverges.

43. $\frac{n^2 - 20}{n^5 + 4} \to 0$ so the series may converge or may diverge. (Later we will see that it converges.)

Section 10.3

1. $\sum_{k=0}^{\infty} (\frac{1}{3})^k = \frac{1}{1-(\frac{1}{3})} = \frac{3}{2}$

2. $\sum_{k=0}^{\infty} (\frac{2}{3})^k = \frac{1}{1-(\frac{2}{3})} = 3$

3. $\frac{1}{8} \cdot \sum_{k=0}^{\infty} (\frac{1}{2})^k = \frac{1}{8} \cdot \frac{1}{1-(\frac{1}{2})} = \frac{1}{4}$

5. $-\frac{2}{3} \cdot \sum_{k=0}^{\infty} (-\frac{2}{3})^k = -\frac{2}{3} \cdot \frac{1}{1-(-\frac{2}{3})} = -\frac{2}{5}$

7. (a) $\frac{1}{2} \cdot \sum_{k=0}^{\infty} (\frac{1}{2})^k = \frac{1}{2} \cdot \frac{1}{1-(\frac{1}{2})} = \frac{1}{2} \cdot \frac{2}{1} = 1$, $\quad \frac{1}{3} \cdot \sum_{k=0}^{\infty} (\frac{1}{3})^k = \frac{1}{3} \cdot \frac{1}{1-(\frac{1}{3})} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$

(b) $\frac{1}{a} \cdot \sum_{k=0}^{\infty} (\frac{1}{a})^k = \frac{1}{a} \cdot \frac{1}{1-(\frac{1}{a})} = \frac{1}{a} \cdot \frac{a}{a-1} = \frac{1}{a-1}$

9. (a) $40 \cdot (0.4)^{n-1}$ (b) $40 \cdot \sum_{k=0}^{\infty} (0.4)^k$ (c) $40 \cdot \frac{1}{1-0.4} = \frac{40}{0.6} = 66\frac{2}{3}$ ft.

11. (a) $\sum_{k=1}^{\infty} (\frac{1}{2})^k$ (b) $\frac{1}{2}$, $\frac{1}{4}$, $(\frac{1}{2})^n$ (c) All of the cake.

13. $1 + \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \ldots = \sum_{k=0}^{\infty} (\frac{1}{4})^k = \frac{4}{3}$.

15. (a) Area $= 1 + \frac{3}{9} + \frac{3}{9} \cdot \frac{4}{9} + \frac{3}{9}(\frac{4}{9})^2 + \frac{3}{9}(\frac{4}{9})^3 + \ldots = 1 + \frac{3}{9} \left\{ 1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3 + \ldots \right\}$

$$= 1 + \frac{3}{9} \left\{ \frac{1}{1-(4/9)} \right\} = 1 + \frac{3}{5} = \frac{8}{5} = 1.6 .$$

(b) Let L be the length of the original triangle ($L = 3\sqrt{\frac{4}{\sqrt{3}}}$) and P_n be the perimeter at the n^{th} step. Then $P_0 = 3L$. $P_1 = 3 \cdot 4 \cdot (\frac{L}{3}) = 4L$,

$P_2 = 3 \cdot 4^2 \cdot (\frac{L}{3^2}) = 3L(\frac{4}{3})^2$

$P_3 = 3 \cdot 4^3 \cdot (\frac{L}{3^3}) = 3L(\frac{4}{3})^3$

$P_4 = 3 \cdot 4^4 \cdot (\frac{L}{3^4}) = 3L(\frac{4}{3})^4$, and, in general,

$P_n = 3 \cdot 4^n \cdot (\frac{L}{3^n}) = 3L(\frac{4}{3})^n$.

Since $\frac{4}{3} > 1$, the sequence of terms $3L(\frac{4}{3})^n$ grows without bound, and the perimeter "approaches infinity."

17. (a) Height $= 2 + 2(\frac{1}{2}) + 2(\frac{1}{4}) + 2(\frac{1}{8}) + \ldots = 2\{1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots\} = 2 \cdot \frac{1}{1-\frac{1}{2}} = 4.$

(b) Surface area $= 4\pi(1)^2 + 4\pi(\frac{1}{2})^2 + 4\pi(\frac{1}{4})^2 + 4\pi(\frac{1}{8})^2 + \ldots$

$$= 4\pi\{1 + \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \ldots\} = 4\pi \cdot \frac{1}{1-\frac{1}{4}} = \frac{16\pi}{3} \approx 16.755 .$$

(c) Volume $= \frac{4\pi}{3}(1)^3 + \frac{4\pi}{3}(\frac{1}{2})^3 + \frac{4\pi}{3}(\frac{1}{4})^3 + \frac{4\pi}{3}(\frac{1}{8})^3 + \ldots$

$$= \frac{4\pi}{3}\{1 + \frac{1}{8} + (\frac{1}{8})^2 + (\frac{1}{8})^3 + \ldots\} = \frac{4\pi}{3} \cdot \frac{1}{1-\frac{1}{8}} = \frac{32\pi}{21} \approx 4.787 .$$

19. $0.8888\ldots = \frac{8}{10} + \frac{8}{10^2} + \frac{8}{10^3} + \ldots = \frac{8}{10}\{1 + \frac{1}{10} + (\frac{1}{10})^2 + (\frac{1}{10})^3 + \ldots\} = \frac{8}{10}\{\frac{10}{9}\} = \frac{8}{9}$.

$0.9999\ldots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \ldots = \frac{9}{10}\{1 + \frac{1}{10} + (\frac{1}{10})^2 + (\frac{1}{10})^3 + \ldots\} = \frac{9}{10}\{\frac{10}{9}\} = 1$.

$0.285714\ldots = \frac{285714}{1000000} + \frac{285714}{1000000^2} + \frac{285714}{1000000^3} + \ldots$

$$= \frac{285714}{1000000}\{1 + \frac{1}{1000000} + (\frac{1}{1000000})^2 + (\frac{1}{1000000})^3 + \ldots\}$$

$$= \frac{285714}{1000000}\{\frac{1000000}{999999}\} = \frac{285714}{999999} .$$

21. Series converges for $|2x + 1| < 1$: $-1 < x < 0$.

23. Series converges for $|1 - 2x| < 1$: $0 < x < 1$.

25. Series converges for $|7x| < 1$: $-\frac{1}{7} < x < \frac{1}{7}$.

27. Series converges for $|\frac{x}{2}| < 1$: $-2 < x < 2$.

29. Series converges for $|2x| < 1$: $-\frac{1}{2} < x < \frac{1}{2}$.

31. Series converges for $|\sin(x)| < 1$: for all $x \neq \frac{\pi}{2} \pm N\pi$ for integer values of N.

33. The formula is correct if $|x| < 1$. The value $x = 2$ does not satisfy the condition $|x| < 1$, so the formula does not apply.

35. $s_4 = (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) = \frac{1}{3} - \frac{1}{5}$, $s_5 = \frac{1}{3} - \frac{1}{6}$, $s_n = \frac{1}{3} - \frac{1}{n+1} \to \frac{1}{3}$

37. $s_3 = (1^3 - 2^3) + (2^3 - 3^3) + (3^3 - 4^3) = 1 - 4^3$, $s_4 = 1 - 5^3$, $s_n = 1 - (n+1)^3 \to -\infty$.

39. $s_4 = (f(3) - f(4)) + (f(4) - f(5)) = f(3) - f(5)$, $s_5 = f(3) - f(6)$, $s_n = f(3) - f(n+1)$

41. $s_4 = \sin(1) - \sin(\frac{1}{5}) \approx 0.643$, $s_5 = \sin(1) - \sin(\frac{1}{6}) \approx 0.676$, $s_n = \sin(1) - \sin(\frac{1}{n+1}) \to \sin(1) \approx 0.841$.

43. $s_4 = (\frac{1}{2^2} - \frac{1}{3^2}) + (\frac{1}{3^2} - \frac{1}{4^2}) + (\frac{1}{4^2} - \frac{1}{5^2}) = \frac{1}{4} - \frac{1}{25}$, $s_5 = \frac{1}{4} - \frac{1}{36}$, $s_n = \frac{1}{4} - \frac{1}{(n+1)^2} \to \frac{1}{4}$

45. & 47. On your own.

Section 10.3.5

Integrals and sums

1. The sum.

2. The integral.

3. $\sum_{k=1}^{\infty} f(k)$

4. $\sum_{k=1}^{\infty} f(k)$

5. $\sum_{k=2}^{\infty} f(k)$

6. (b) $f(1) + f(2)$

7. (b) $f(1) + f(2)$

8. (d) $f(3) + f(4)$

9. (c) $f(2) + f(3)$

10. (b), (d), (a), (c) : $\int_2^4 f(x)\,dx < f(2) + f(3) < \int_1^3 f(x)\,dx < f(1) + f(2)$

11. (c), (a), (d), (b) : $f(1) + f(2) + f(3) < \int_1^4 f(x)\,dx < f(2) + f(3) + f(4) < \int_2^5 f(x)\,dx$

Comparisons

12. (a) No. You are definitely too tall.
    (b) Apply. You **may** meet the requirements.
    (c) No. Definitely too short.
    (d) Apply. You **may** meet the requirements.
    (e) You do not meet the requirements if you are {shorter than Sam} or {taller than Tom}.
    (f) You do not have enough information if you are {shorter than Tom} or {taller than Sam}.

13. (a) You did well (better than Wendy).
    (b) You may have done well or poorly.
    (c) You may have done well or poorly.
    (d) You did poorly (worse than Paula).

14. (a) Baker is too easy (easier than Index).
    (b) Baker may be right for you.
    (c) Baker may be right for you.
    (d) Baker is too hard for you (harder than Liberty Bell).
    (e) Baker may be a good climb for you if it is harder than Index and easier than Liberty Bell.
    (f) Baker is too easy if Baker is easier than Index. Baker is too hard if Baker is harder than Liberty Bell.

15. (a) Expect Unknown to be (very) good.
    (b) Unknown is still unknown.
    (c) Unknown is still unknown.
    (d) Unknown is bad.

16. $\sum_{k=1}^{\infty} \frac{1}{k^2+1} < \sum_{k=1}^{\infty} \frac{1}{k^2}$

17. $\sum_{k=2}^{\infty} \frac{1}{k^3-5} > \sum_{k=2}^{\infty} \frac{1}{k^3}$

18. $\sum_{k=1}^{\infty} \frac{1}{k^2+3k-1} < \sum_{k=1}^{\infty} \frac{1}{k^2}$

19. $\sum_{k=3}^{\infty} \frac{1}{k^2+5k} > \sum_{k=3}^{\infty} \frac{1}{k^3+k-1}$

Ratios of successive terms

20. $a_k = 3k$, $a_{k+1} = 3(k+1)$, $\frac{a_{k+1}}{a_k} = \frac{k+1}{k}$.

21. $a_k = k + 3$, $a_{k+1} = (k+1) + 3$, $\frac{a_{k+1}}{a_k} = \frac{k+4}{k+3}$.

22. $a_k = 2k + 5$, $a_{k+1} = 2(k+1) + 5$, $\frac{a_{k+1}}{a_k} = \frac{2k+7}{2k+5}$.

23. $a_k = 3/k$, $a_{k+1} = \frac{3}{k+1}$, $\frac{a_{k+1}}{a_k} = \frac{\frac{3}{k+1}}{\frac{3}{k}} = \frac{k}{k+1}$.

24. $a_k = k^2$, $a_{k+1} = (k+1)^2$, $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{k^2}$. 25. $a_k = 2^k$, $a_{k+1} = 2^{k+1}$, $\frac{a_{k+1}}{a_k} = \frac{2^{k+1}}{2^k} = 2$.

26. $a_k = (1/2)^k$, $a_{k+1} = (1/2)^{k+1}$, $\frac{a_{k+1}}{a_k} = \frac{(1/2)^{k+1}}{(1/2)^k} = \frac{1}{2}$.

27. $a_k = x^k$, $a_{k+1} = x^{k+1}$, $\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{x^k} = x$.

28. $a_k = (x - 1)^k$, $a_{k+1} = (x-1)^{k+1}$, $\frac{a_{k+1}}{a_k} = \frac{(x-1)^{k+1}}{(x-1)^k} = x - 1$.

29. $\sum_{k=1}^{\infty} (\frac{1}{2})^k$ is a geometric series with $r = 1/2$ so the series converges. $\frac{a_{k+1}}{a_k} = \frac{(1/2)^{k+1}}{(1/2)^k} = \frac{1}{2}$.

30. $\sum_{k=1}^{\infty} (\frac{1}{5})^k$ is a geometric series with $r = 1/5$ so the series converges. $\frac{a_{k+1}}{a_k} = \frac{(1/5)^{k+1}}{(1/5)^k} = \frac{1}{5}$.

31. $\sum_{k=1}^{\infty} 2^k$ is a geometric series with $r = 2$ so the series diverges. $\frac{a_{k+1}}{a_k} = \frac{2^{k+1}}{2^k} = 2$.

32. $\sum_{k=1}^{\infty} (-3)^k$ is a geometric series with $r = -3$ so the series diverges. $\frac{a_{k+1}}{a_k} = \frac{(-3)^{k+1}}{(-3)^k} = -3$.

33. $\sum_{k=1}^{\infty} 4 = 4 + 4 + 4 + \ldots$ diverges by the N^{th} Term Test for Divergence since $a_n = 4$ for all n, and

$a_n = 4$ does not approach 0. $\frac{a_{k+1}}{a_k} = \frac{4}{4} = 1$.

34. $\sum_{k=1}^{\infty} (-1)^k$ diverges by the N^{th} Term Test for Divergence since a_n does not approach 0. (It also is a

geometric series with $r = -1$ and $| r | = 1$.) $\frac{a_{k+1}}{a_k} = \frac{(-1)^{k+1}}{(-1)^k} = -1$.

35. $\sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series which diverges. $\frac{a_{k+1}}{a_k} = \frac{\frac{1}{k+1}}{\frac{1}{k}} = \frac{k}{k+1}$.

36. $\sum_{k=1}^{\infty} \frac{7}{k} = 7 \cdot \sum_{k=1}^{\infty} \frac{1}{k}$ is the divergent harmonic series. $\frac{a_{k+1}}{a_k} = \frac{\frac{7}{k+1}}{\frac{7}{k}} = \frac{k}{k+1}$.

Alternating terms

37. If $a_5 > 0$, then $s_4 < s_5$.

38. If $a_5 = 0$, then $s_4 = s_5$.

39. If $a_5 < 0$, then $s_4 > s_5$.

40. If $a_{n+1} > 0$ for all n, then $s_n < s_{n+1}$ for all n.

41. If $a_{n+1} < 0$ for all n, then $s_n > s_{n+1}$ for all n.

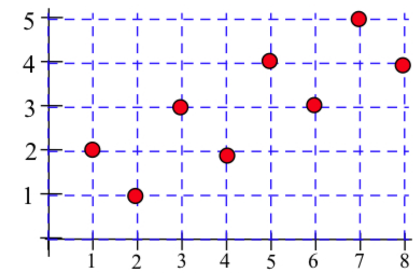
42. If $a_4 > 0$ and $a_5 < 0$, then $s_3 < s_4$ and $s_4 > s_5$.

43. If $a_4 = 0.2$ and $a_5 = -0.1$ and $a_6 = 0.2$, then $s_3 < s_5 < s_4$.

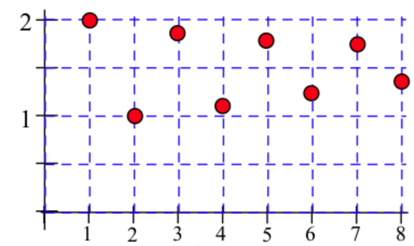
44. If $a_4 = -0.3$ and $a_5 = 0.2$ and $a_6 = -0.1$, then $s_3 > s_5 > s_4$.

45. If $a_4 = -0.3$ and $a_5 = -0.2$ and $a_6 = 0.1$, then $s_3 > s_4 > s_5$.

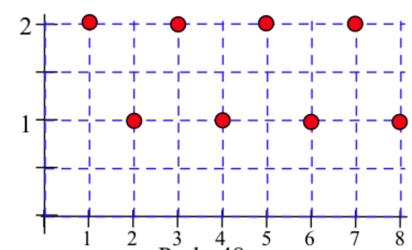
46. $s_1 = 2, s_2 = 1, s_3 = 3, s_4 = 2, s_5 = 4, s_6 = 3, s_7 = 5, s_8 = 4$.


Prob. 46

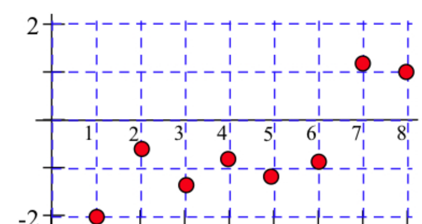
47. $s_1 = 2, s_2 = 1, s_3 = 1.9, s_4 = 1.1, s_5 = 1.8, s_6 = 1.2,$
$s_7 = 1.7, s_8 = 1.3$. The graph is given.


Prob. 47

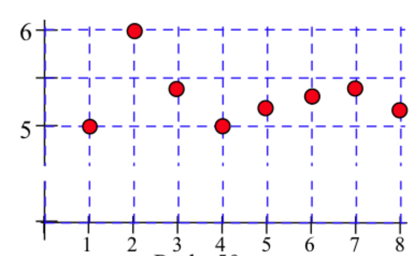
48. $s_1 = 2, s_2 = 1, s_3 = 2, s_4 = 1, s_5 = 2, s_6 = 1, s_7 = 2, s_8 = 1$.
The graph is given.


Prob. 48

49. $s_1 = -2, s_2 = -0.5, s_3 = -1.3, s_4 = -0.7, s_5 = -1.1$, $s_6 = -0.9,$
$s_7 = 1.1, s_8 = 1.0$. The graph is given.


Prob. 49

50. $s_1 = 5, s_2 = 6, s_3 = 5.4, s_4 = 5.0$, $s_5 = 5.2$, $s_6 = 5.3$,
$s_7 = 5.4$, $s_8 = 5.2$. The graph is given.


Prob. 50

51. If the a_k alternate in sign, then the graph of the partial sums s_n follows an "up–down–up–down" pattern?

52. If the a_k alternate in sign and decrease in magnitude (the $| a_k |$ is decreasing), then the graph of the partial sums s_n forms a "narrowing funnel" pattern.

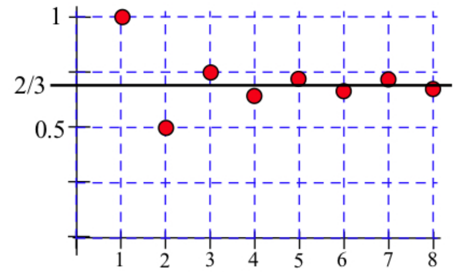
53. (a) The terms a_k alternate in sign for the graphs D, E, and F.
(b) The $| a_k |$ decrease for the graphs A and D.
(c) The terms a_k alternate in sign and decrease in absolute value for the graph D.

54. (a) The terms a_k alternate in sign for the graphs C, D, E, and F in Fig. 14.
(b) The $| a_k |$ decrease for the graphs B and C.
(c) The terms a_k alternate in sign and decrease in absolute value for the graph C.

55. The graph of the first eight partial sums of

$$\sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k = 1 - \frac{1}{2} + \frac{1}{4} - \ldots = \frac{2}{3} \quad \text{is given.}$$

Notice the "narrowing funnel" shape of the graph.



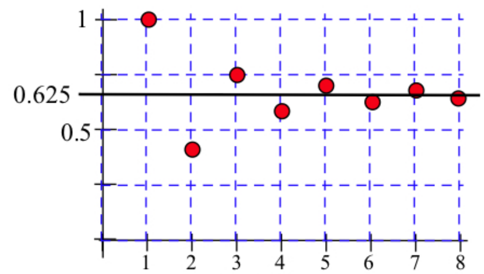
Prob. 55

56. The graph of the first eight partial sums of

$$\sum_{k=0}^{\infty} (-0.6)^k = 1 - 0.6 + 0.36 - \ldots = 0.625$$

is given.

Notice the "narrowing funnel" shape of the graph.



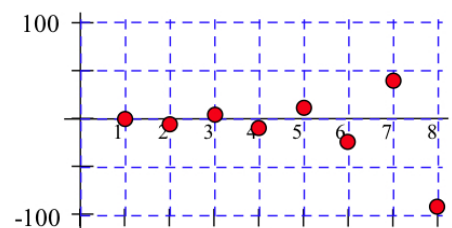
Prob. 56

57. The graph is given of the first eight partial sums of the divergent series

$$\sum_{k=0}^{\infty} (-2)^k = 1 - 2 + 4 - \ldots$$

This is a divergent series (N^{th} Term Test for Divergence).

Notice the "widening funnel" shape of the graph.



Prob. 57

Section 10.4 (Odd numbered problem solutions followed by even numbered problem answers.)

1. $\displaystyle\int_1^{\infty} \frac{1}{2x+5}\,dx = \frac{1}{2}\ln|2x+5|\Big|_1^A = \frac{1}{2}\ln|2A+5| - \frac{1}{2}\ln|7| \to \infty$ (as $A \to \infty$) so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k+5}$ diverges.

3. $\displaystyle\int_1^{\infty} (2x+5)^{-3/2}\,dx = -(2x+5)^{-1/2}\Big|_1^A = \frac{-1}{\sqrt{2A+5}} - \frac{-1}{\sqrt{7}} \to \frac{1}{\sqrt{7}}$ (as $A \varnothing \infty$) so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k+5)^{3/2}}$ converges.

5. $\displaystyle\int \frac{1}{x \cdot (\ln(x))^2}\,dx = \frac{-1}{\ln(x)} + C$ (using a u–substitution with $u = \ln(x)$ and $du = \frac{1}{x}\,dx$) so

$\displaystyle\int_2^{\infty} \frac{1}{x \cdot (\ln(x))^2}\,dx = \frac{-1}{\ln(x)}\Big|_2^A = \frac{-1}{\ln(A)} - \frac{-1}{\ln(2)} \to \frac{1}{\ln(2)}$ (as $A \varnothing \infty$) so $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \cdot (\ln(k))^2}$ converges.

7. $\displaystyle\int_1^{\infty} \frac{1}{x^2+1}\,dx = \arctan(x)\Big|_1^A = \arctan(A) - \arctan(1) \to \frac{\pi}{2} - \frac{\pi}{4}$ (as $A \varnothing \infty$) so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ converges.

9. This is a telescoping series:

$$\sum_{k=1}^{\infty} \left\{ \frac{1}{k} - \frac{1}{k+3} \right\} = \left\{ 1 - \frac{1}{4} \right\} + \left\{ \frac{1}{2} - \frac{1}{5} \right\} + \left\{ \frac{1}{3} - \frac{1}{6} \right\} + \left\{ \frac{1}{4} - \frac{1}{7} \right\} + \left\{ \frac{1}{5} - \frac{1}{8} \right\} + \ldots \to 1 + \frac{1}{2} + \frac{1}{3} .$$

The Integral Test also works:

$$\int_1^{\infty} \frac{1}{x} - \frac{1}{x+3}\, dx = \ln|x| - \ln|x+3| \Big|_1^A = \{ \ln|A| - \ln|A+3| \} - \{ \ln(1) - \ln(4) \}$$

$$= \ln\left| \frac{4A}{A+3} \right| \to \ln(4) \quad (\text{as } A \to \infty) \text{ so } \sum_{k=1}^{\infty} \left\{ \frac{1}{k} - \frac{1}{k+3} \right\} \text{ converges.}$$

(Notice that the "telescoping series" method gives the value of the series, but the Integral Test only tells us that the series converges. For this series, the "telescoping series" method is both easier and more precise.)

11. $\displaystyle\int_1^{\infty} \frac{1}{x(x+5)}\, dx = \int_1^{\infty} \frac{1}{5} \left\{ \frac{1}{x} - \frac{1}{x+5} \right\} dx$ (using the method of Partial Fraction Decomposition)

$$= \frac{1}{5} \{ \ln|x| - \ln|x+5| \} = \frac{1}{5} \left\{ \ln\left| \frac{x}{x+5} \right| \right\} \Big|_1^A = \frac{1}{5} \left\{ \ln\left| \frac{A}{A+5} \right| - \ln\left| \frac{1}{6} \right| \right\} \to \frac{1}{5} \left\{ \ln(1) - \ln\left(\frac{1}{6}\right) \right\}$$

(as $A \to \infty$) so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\bullet(k+5)}$ converges.

13. $\displaystyle\int_1^{\infty} x \bullet e^{-(x^2)}\, dx = \frac{-1}{2} e^{-(x^2)} \Big|_1^A = \left(\frac{-1}{2} e^{-(A^2)} \right) - \left(\frac{-1}{2} e^{-1} \right) = \frac{1}{2e} - \frac{1}{2e^{(A^2)}} \to \frac{1}{2e}$ (as $A \varnothing \infty$) so

$\displaystyle\sum_{k=1}^{\infty} k \bullet e^{-(k^2)}$ converges.

15. $\displaystyle\int \frac{1}{\sqrt{6x+10}}\, dx = \frac{1}{3}\sqrt{6x+10} + C$ (using a u–substitution with $u = 6x + 10$ and $du = 6\, dx$). Then

$$\int_1^{\infty} \frac{1}{\sqrt{6x+10}}\, dx = \frac{1}{3}\sqrt{6x+10} \Big|_1^A = \frac{1}{3}\sqrt{6A+10} - \frac{1}{3}\sqrt{16} \to \infty \quad (\text{as } A \to \infty)$$

so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{6k+10}}$ diverges.

17. $p = 3 > 1$ so $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges.

19. $p = 1/2 < 1$ so $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

21. $p = 3/2 > 1$ so $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k^{3/2}}$ converges.

23. $\sum_{k=1}^{\infty} \frac{1}{k^3}$: $\int_1^{11} \frac{1}{x^3}\,dx = 0.4958677$, $1 + \int_1^{10} \frac{1}{x^3}\,dx = 1.495$ so $0.4958677 < s_{10} < 1.495$.

$\int_1^{101} \frac{1}{x^3}\,dx = 0.499951$, $1 + \int_1^{100} \frac{1}{x^3}\,dx = 1.49995$ so $0.499951 < s_{100} < 1.49995$.

$\int_1^{1000001} \frac{1}{x^3}\,dx = 0.5000000$, $1 + \int_1^{1000000} \frac{1}{x^3}\,dx = 1.5000000$ so $0.5000000 < s_{1000000} < 1.5000000$.

25. $\sum_{k=1}^{\infty} \frac{1}{k + 1000}$: $\int_1^{11} \frac{1}{x + 1000}\,dx = 0.0099404$, $\frac{1}{1001} + \int_1^{10} \frac{1}{x + 1000}\,dx = 0.0099498$

so $\mathbf{0.00994}04 < s_{10} < \mathbf{0.00994}98$. (This is a very precise estimate of s_{10} .)

$\int_1^{101} \frac{1}{x + 1000}\,dx = 0.09522$, $\frac{1}{1001} + \int_1^{100} \frac{1}{x + 1000}\,dx = 0.0953$ so $\mathbf{0.095}22 < s_{100} < \mathbf{0.095}3$.

$\int_1^{1000001} \frac{1}{x + 1000}\,dx = 6.90776$, $\frac{1}{1001} + \int_1^{1000000} \frac{1}{x + 1000}\,dx = 6.90875$ so $\mathbf{6.90}776 < s_{1000000} < \mathbf{6.90}875$.

27. $\sum_{k=1}^{\infty} \frac{1}{k^2 + 100}$: $\int_1^{11} \frac{1}{x^2 + 100}\,dx = \frac{1}{10} \arctan(\frac{x}{10})\Big|_1^{11} = 0.0733$, $\frac{1}{101} + \int_1^{10} \frac{1}{x^2 + 100}\,dx = 0.0783$

so $\mathbf{0.07}3 < s_{10} < \mathbf{0.07}8$. Also, $\mathbf{0.1}37 < s_{100} < \mathbf{0.1}47$ and $\mathbf{0.1}471 < s_{1000000} < \mathbf{0.1}57$.

29. For $\mathbf{q \neq 1}$, let $u = \ln(x)$ and $du = \frac{1}{x}\,dx$.

Then $\int \frac{1}{x\cdot(\ln(x))^q}\,dx = \int \frac{1}{(u)^q}\,du = \frac{1}{1-q} u^{-q+1} = \frac{1}{1-q} (\ln(x))^{1-q} + C$.

Then $\int_2^{\infty} \frac{1}{x\cdot(\ln(x))^q}\,dx = \frac{1}{1-q} (\ln(x))^{1-q} \Big|_2^{A} = \frac{1}{1-q} (\ln(A))^{1-q} - \frac{1}{1-q} (\ln(2))^{1-q}$.

If $q < 1$, then $\frac{1}{1-q} (\ln(A))^{1-q} - \frac{1}{1-q} (\ln(2))^{1-q} \to \infty$ (as $A \to \infty$) so $\sum_{k=2}^{\infty} \frac{1}{k\cdot(\ln k)^q}$ diverges.

If $q > 1$, then $\frac{1}{1-q} (\ln(A))^{1-q} - \frac{1}{1-q} (\ln(2))^{1-q} \to -\frac{1}{1-q} (\ln(2))^{1-q}$ (as $A \to \infty$)

so $\sum_{k=2}^{\infty} \frac{1}{k\cdot(\ln k)^q}$ converges.

If $\mathbf{q = 1}$, then

$$\int_2^\infty \frac{1}{x\cdot(\ln(x))^q}\,dx = \int_2^\infty \frac{1}{x\cdot\ln(x)}\,dx = \ln|\ln(x)|\Big|_2^A = \ln|\ln(A)| - \ln|\ln(2)| \to \infty \text{ (as } A \to \infty)$$

so $\sum_{k=2}^{\infty} \frac{1}{k\cdot(\ln k)}$ diverges.

"Q–Test:" $\sum_{k=2}^{\infty} \frac{1}{k\cdot(\ln k)^q}$ $\begin{cases} \text{diverges} & \text{if } q \le 1 \\ \text{converges} & \text{if } q > 1 \end{cases}$

31. $q = 3 > 1$ so $\sum_{k=2}^{\infty} \frac{1}{k\cdot(\ln k)^3}$ converges.

33. $\sum_{k=2}^{\infty} \frac{1}{k\cdot\ln(k^3)} = \sum_{k=2}^{\infty} \frac{1}{3k\cdot\ln(k)} = \frac{1}{3}\sum_{k=2}^{\infty} \frac{1}{k\cdot\ln(k)}$ which diverges ($q = 1$).

Section 10.4 Some Even Answers

2. Converge	4. Diverge	6. Converge	8. Converge	10. Converge
12. Converge	14. Converge	16. Converge	18. Diverge	20. Diverge
30. Diverge	32. Diverge			

Section 10.5 (Odd numbered problem solutions followed by even numbered problem answers.)

1. $\sum_{k=1}^{\infty} \frac{\cos^2(k)}{k^2} \le \sum_{k=1}^{\infty} \frac{1}{k^2}$ which converges by the P–Test (p=2) so $\sum_{k=1}^{\infty} \frac{\cos^2(k)}{k^2}$ converges.

3. $\sum_{n=3}^{\infty} \frac{5}{n-1} > 5\sum_{n=3}^{\infty} \frac{1}{n}$ which is the harmonic series and is divergent, so $\sum_{n=3}^{\infty} \frac{5}{n-1}$ diverges.

5. $-1 \le \cos(x) \le 1$ so $2 \le 3 + \cos(x) \le 4$. Then $\sum_{j=1}^{\infty} \frac{3+\cos(j)}{j} > 2\sum_{j=1}^{\infty} \frac{1}{j}$ which is the harmonic series and is divergent, so $\sum_{j=1}^{\infty} \frac{3+\cos(j)}{j}$ diverges.

7. $\sum_{k=1}^{\infty} \frac{\ln(k)}{k} > \sum_{k=1}^{\infty} \frac{1}{k}$ which is the harmonic series and is divergent, so $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$ diverges.

9. $\sum_{k=1}^{\infty} \frac{k+9}{k\cdot 2^k} = \sum_{k=1}^{\infty} \frac{k+9}{k}\cdot\frac{1}{2^k} < 10\sum_{k=1}^{\infty} \frac{1}{2^k}$ which is a convergent geometric series ($r = \frac{1}{2}$)

so $\sum_{k=1}^{\infty} \frac{k+9}{k\cdot 2^k}$ converges.

11. $\sum_{n=1}^{\infty} \frac{1}{1+2+3+\ldots+(n-1)+n} = \sum_{n=1}^{\infty} \frac{1}{\frac{n(n+1)}{2}} = \sum_{n=1}^{\infty} \frac{2}{n(n+1)}\ ) < 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges by

the P–Test (p = 2) so $\sum_{n=1}^{\infty} \frac{1}{1+2+3+\ldots+(n-1)+n}$ converges.

13. Let $a_k = \frac{k+1}{k^2+4}$ and $b_k = \frac{1}{k}$. Then $\frac{a_k}{b_k} = \frac{k^2+k}{k^2+4} \to 1$ and $\sum_{k=3}^{\infty} \frac{1}{k}$ diverges so $\sum_{k=3}^{\infty} \frac{k+1}{k^2+4}$ diverges.

15. Let $a_w = \frac{5}{w+1}$ and $b_w = \frac{5}{w}$. Then $\frac{a_w}{b_w} = \frac{w}{w+1} \to 1$ and $\sum_{w=1}^{\infty} \frac{1}{w}$ diverges so $\sum_{w=1}^{\infty} \frac{5}{w+1}$ diverges.

17. $\sum_{k=1}^{\infty} \frac{k^3}{(1+k^2)^3} = \sum_{k=1}^{\infty} \left(\frac{k}{1+k^2}\right)^3$. Let $a_k = \left(\frac{k}{1+k^2}\right)^3$ and $b_k = \frac{1}{k^3}$.

Then $\frac{a_k}{b_k} = \left(\frac{k}{1+k^2}\right)^3 \cdot \frac{k^3}{1} = \left(\frac{k^2}{1+k^2}\right)^3 \to (1)^3 = 1$ and $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges by the

P–Test so $\sum_{k=1}^{\infty} \frac{k^3}{(1+k^2)^3}$ converges.

19. $\sum_{n=1}^{\infty} \left(\frac{5-\frac{1}{n}}{n}\right)^3 = \sum_{n=1}^{\infty} \left(\frac{5}{n} - \frac{1}{n^2}\right)^3 = \sum_{n=1}^{\infty} \left(\frac{5n-1}{n^2}\right)^3$. Let $a_n = \left(\frac{5n-1}{n^2}\right)^3$ and $b_n = \frac{1}{n^3}$.

Then $\frac{a_n}{b_n} = \left(\frac{5n-1}{n^2}\right)^3 \frac{n^3}{1} = \left(\frac{5n-1}{n^2} \cdot \frac{n}{1}\right)^3 = \left(\frac{5n-1}{n}\right)^3 \to 5^3 = 125$ (positive and finite).

$\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges by the P–Test so $\sum_{n=1}^{\infty} \left(\frac{5-\frac{1}{n}}{n}\right)^3$ converges.

21. $\sum_{j=1}^{\infty} \left(1 - \frac{1}{j}\right)^j$ diverges by the N^{th} Term Test for Divergence since $a_j = \left(1 - \frac{1}{j}\right)^j \to e^{-1} \neq 0$ as $j \to \infty$.

We could use the Limit Comparison Test by taking $b_j = e^{-1}$ and showing that $\frac{a_j}{b_j} \to 1$, but the N^{th} Term Test for Divergence is more direct for this series.

23. $\sum_{k=1}^{\infty} \frac{7k}{\sqrt{k^3+5}}$: The dominant term series is $\sum_{k=1}^{\infty} \frac{k}{k^{3/2}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ which diverges by the P–Test (p=1/2).

25. $\sum_{j=1}^{\infty} \frac{j^3 - 4j + 3}{2j^4 + 7j^6 + 9}$: The dominant term series is $\sum_{j=1}^{\infty} \frac{j^3}{j^6} = \sum_{j=1}^{\infty} \frac{1}{j^3}$ which converges by the P–Test (p=3).

27. $\sum_{n=1}^{\infty} \left(\frac{\arctan(3n)}{2n} \right)^2$: The dominant term series is $\sum_{n=1}^{\infty} \left(\frac{\pi/2}{2n} \right)^2 = \frac{\pi^2}{16} \sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges by the P–Test (p=2).

29. $\sum_{j=1}^{\infty} \frac{\sqrt{j^3 + 4j^2}}{j^2 + 3j - 2}$: The dominant term series is $\sum_{j=1}^{\infty} \frac{j^{3/2}}{j^2} = \sum_{j=1}^{\infty} \frac{1}{j^{1/2}}$ which diverges by the P–Test.

31. $\sum_{n=2}^{\infty} \frac{n^2 + 10}{n^3 - 2}$ diverges using dominant terms and the P–Test (p=1).

33. $\sum_{k=1}^{\infty} \frac{3}{2k + 1}$ diverges using dominant terms and the P–Test (p=1).

35. $\sum_{n=1}^{\infty} \frac{2n^3 + n^2 + 5}{(3 + n^2)^2}$ diverges using dominant terms and the P–Test (p=1).

37. $\sum_{k=1}^{\infty} \left(\frac{1 - \frac{2}{k}}{k} \right)^3 = \sum_{k=1}^{\infty} \left(\frac{k-2}{k^2} \right)^3$ converges using dominant terms $\left(\frac{k}{k^2} \right)^3 = \left(\frac{1}{k} \right)^3$ and the P–Test (p=3).

39. $\sum_{k=1}^{\infty} \frac{k + 5}{k \cdot 3^k}$ converges by using dominant terms $\frac{k}{k \cdot 3^k} = \frac{1}{3^k} = \left(\frac{1}{3} \right)^k$ and the Geometric Series Test (r = 1/3).

41. $\sum_{k=1}^{\infty} \frac{k + 2}{\sqrt{k^2 + 1}}$ diverges using dominant terms and the N^{th} Term Test for Divergence.

43. $\sum_{j=1}^{\infty} \frac{3}{e^j + j}$ converges using dominant terms $\frac{1}{e^j} = \left(\frac{1}{e} \right)^j$ and the Geometric Series Test (r = 1/e).

45. $\sum_{n=1}^{\infty} \left(\frac{\tan(3)}{2 + n} \right)^2$ converges by using dominant terms $\frac{1}{n^2}$ and the P–Test (p=2).

47. $\sum_{k=1}^{\infty} \sin^3\left(\frac{1}{n} \right)$ converges by comparison with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^3}$

49. $\sum_{j=1}^{\infty} \cos^3(\frac{1}{n})$ diverges using the N^{th} Term Test for Divergence: the terms approach $1 \neq 0$.

51. $\sum_{n=1}^{\infty} (1 - \frac{2}{n})^n$ diverges using the N^{th} Term Test for Divergence: the terms approach $\frac{1}{e^2} \neq 0$.

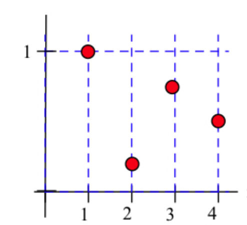
Section 10.5 Some Even Answers

2. Converges	4. Converges	6. Converges	8. Converges	10. Diverges
12. Converges	14. Converges	16. Diverges	18. Converges	20. Diverges
22. Diverges	24. Diverges	26. Converges	28. Converges	30. Converges
32. Converges	34. Converges	36. Converges	38. Converges	40. Converges
42. Converges	44. Converges	46. Converges	48. Diverges	50. Converges

You still need to supply reasons for each answer given below.

R1. Converge	R2. Converge	R3. Diverge	R4. Diverge	R5. Converge
R6. Converge	R7. Converge	R8. Diverge	R9. Converge	R10. Diverge
R11. Converge	R12. Diverge	R13. Diverge	R14. Converge	R15. Converge
R16. Converge	R17. Diverge	R18. Diverge	R19. Diverge	R20. Converge
R21. Diverge				

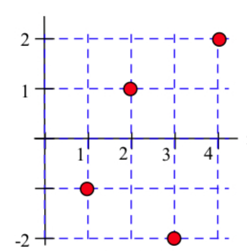
Section 10.6 (Odd numbered problem solutions followed by some even numbered problem answers.)



Prob. 1

1. 1, 0.2, 0.8, 0.4 Alternating (so far). The graph of s_n is shown.

3. –1, 1, –2, 2 Alternating (so far). The graph of s_n is shown.



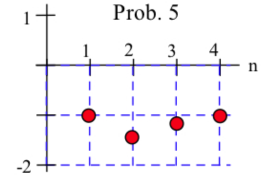
Prob. 3

5. –1, –1.6, –1.2, –1 Not alternating. The graph of s_n is shown.

7. Alternating: $a_1 = 2, a_2 = -1, a_3 = 2, a_4 = -1, a_5 = 2$

9. Not alternating: $a_1 = 2, a_2 = 1, a_3 = -0.9, a_4 = 0.8, a_5 = -0.1$

11. Not alternating: $a_1 = -1, a_2 = 2, a_3 = -1.2, a_4 = 0.2, a_5 = 0.2$

Prob. 5


13. Graphs A and C are not the graphs of partial sums of alternating series.

15. Graph B is not the graph of an alternating series.

17. Converges. 19. Converges. 21. Diverges. 23. Converges.

25. Converges. 27. Diverges. 29. Converges. 31. Converges.

33. $s_4 = -\frac{1}{\ln(2)} + \frac{1}{\ln(3)} - \frac{1}{\ln(4)} + \frac{1}{\ln(5)} \approx -0.63247$.

$| s_4 - S | < \frac{1}{\ln(6)} \approx 0.55811$. $-1.19058 < S < -0.07436$

35. $s_4 = (-0.8)^2 + (-0.8)^3 + (-0.8)^4 + (-0.8)^5 \approx 0.20992$. $| s_4 - S | < (0.8)^6 \approx 0.26214$.

$-0.05222 < S < 0.47206$. (For your information: $s_{10} \approx 0.317378$, $s_{50} \approx 0.355550$, $s_{100} \approx 0.355556$)

37. $s_4 = \sin(1) - \sin(\frac{1}{2}) + \sin(\frac{1}{3}) - \sin(\frac{1}{4}) \approx 0.441836$. $| s_4 - S | < \sin(\frac{1}{5}) \approx 0.198669$.

$0.243167 < S < 0.640505$. ($s_{10} \approx 0.503356$, $s_{50} \approx 0.0.540897$, $s_{100} \approx 0.5458219$)

39. $s_4 = -1 + \frac{1}{8} - \frac{1}{27} + \frac{1}{64} \approx -0.896412$. $| s_4 - S | < \frac{1}{125} = 0.008$. $-0.904412 < S < -0.888412$.

($s_{10} \approx -0.901116$, $s_{50} \approx -0.901539$, $s_{100} \approx -0.901542$)

41. $| \frac{1}{(n+1)+6} | \le 0.01$ so $n = 93$ works. Use s_{93} .

43. $| \frac{1}{\sqrt{n+1}} | \le \frac{1}{100}$ so $\sqrt{n+1} \ge 100$ and $n = 10{,}000 - 1$ works. Use s_{9999} .

45. $| \frac{1}{3^{n+1}} | \le \frac{2}{1000}$ so $3^{n+1} \ge 500$ and $n = 5$ works. Use s_5 .

47. $| \frac{1}{(n+1)^2} | \le \frac{1}{1000}$ so $(n+1)^2 \ge 1000$ and $n = 31$ works. Use s_{31} .

49. $| \frac{1}{(n+1) + \ln(n+1)} | \le \frac{4}{100}$ so $(n+1) + \ln(n+1) \ge 25$. Some "calculator experimentation" shows that $n = 21$ works. $(20+1) + \ln(20+1) \approx 24.04$ so $n = 20$ is too small. $(21+1) + \ln(21+1) \approx 25.09$ so $n = 21$ works. Use s_{21} .

51. (a) $S(0.3) = x - \frac{(0.3)^3}{2\cdot3} + \frac{(0.3)^5}{2\cdot3\cdot4\cdot5} - \frac{(0.3)^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7} + \ldots + (-1)^n \frac{(0.3)^{2n+1}}{(2n+1)!} + \ldots$

(b) $s_3 = (0.3) - \frac{(0.3)^3}{2\cdot3} + \frac{(0.3)^5}{2\cdot3\cdot4\cdot5} \approx 0.29552025$.

(c) $| S - s_3 | \le \frac{(0.3)^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7} \approx 0.000000043$. ($\sin(0.3) \approx 0.295520206661$)

53. (a) $S(0.1) = x - \frac{(0.1)^3}{2\cdot3} + \frac{(0.1)^5}{2\cdot3\cdot4\cdot5} - \frac{(0.1)^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7} + \ldots + (-1)^n \frac{(0.1)^{2n+1}}{(2n+1)!} + \ldots$

(b) $s_3 = (0.1) - \frac{(0.1)^3}{2\cdot3} + \frac{(0.1)^5}{2\cdot3\cdot4\cdot5} \approx 0.099833416667$.

(c) $| S - s_3 | \le \frac{(0.1)^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7} \approx 1.98 \cdot 10^{-11}$ . . ($\sin(0.1) \approx 0.099833416647$)

55. (a) $C(1) = 1 - \frac{1^2}{2} + \frac{1^4}{2\cdot3\cdot4} - \frac{1^6}{2\cdot3\cdot4\cdot5\cdot6} + \ldots + (-1)^n \frac{1^{2n}}{(2n)!} + \ldots = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \ldots + (-1)^n \frac{1^{2n}}{(2n)!} + \ldots$

(b) $s_3 = 1 - \frac{1^2}{2} + \frac{1^4}{2\cdot3\cdot4} \approx 0.5416667$.

(c) $| S - s_3 | \le \frac{(1)^7}{2\cdot3\cdot4\cdot5\cdot6\cdot7} \approx 0.0013889$ . . ($\cos(1) \approx 0.5403023$)

57. (a) $C(-0.2) = 1 - \frac{(-0.2)^2}{2} + \frac{(-0.2)^4}{2\cdot3\cdot4} - \frac{(-0.2)^6}{2\cdot3\cdot4\cdot5\cdot6} + \ldots + (-1)^n \frac{1^{2n}}{(2n)!} + \ldots$

(b) $s_3 = 1 - \frac{(-0.2)^2}{2} + \frac{(-0.2)^4}{2\cdot3\cdot4} \approx 0.98006666667$.

(c) $| S - s_3 | \le \frac{(-0.2)^6}{2\cdot3\cdot4\cdot5\cdot6} \approx 8.9 \cdot 10^{-8}$ .. . . ($\cos(-0.2) \approx 0.980066577841$)

59. (a) $E(-1) = 1 + (-1) + \frac{(-1)^2}{2} + \frac{(-1)^3}{2\cdot3} + \frac{(-1)^4}{2\cdot3\cdot4} + \ldots + \frac{(-1)^n}{n!} + \ldots = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \ldots$

(b) $s_3 = 1 - 1 + \frac{1}{2} = 0.5$.

(c) $| S - s_3 | \le \frac{1}{6} \approx 0.16667$ .. . . ($e^{-1} \approx 0.36787944$)

61. (a) $E(-0.2) = 1 + (-0.2) + \frac{(-0.2)^2}{2} + \frac{(-0.2)^3}{2\cdot3} + \frac{(-0.2)^4}{2\cdot3\cdot4} + \ldots + \frac{(-1)^n}{n!} + \ldots$

(b) $s_3 = 1 + (-0.2) + \frac{(-0.2)^2}{2} = 0.82$ (c) $| S - s_3 | \le \frac{(0.2)^3}{6} \approx 0.0013333$ ... . ($e^{-0.2} \approx 0.8187307$)

Section 10.6 Some Even Answers

2. Alternating (so far)
4. Not alternating
6. Alternating (so far)
8. Alternating (so far)
10. Alternating (so far)
12. Not alternating
14. B and C .
16. A, B, and C
18. Converges.
20. Converges.
22. Converges.
24. Converges (to 0).
26. Diverges.
28. Converges.
30. Converges.
42. $n \approx 2.7 \cdot 10^{43}$
44. n = 26 works.
46. n = 16 works.
48. n = 21 works.

Section 10.7 (Odd numbered problem solutions followed by some even numbered problem answers.)

1. Conditionally convergent
3. Absolutely convergent
5. Absolutely convergent
7. Absolutely convergent
9. Conditionally convergent
11. Conditionally convergent
13. Conditionally convergent
15. Absolutely convergent
17. Divergent
19. Conditionally convergent
21. Divergent
23. Absolutely convergent
25. Conditionally convergent
27. Divergent
29. Absolutely convergent

31. $\frac{n!}{(n+1)!} = \frac{1\cdot 2\cdot 3 \ldots \cdot n}{1\cdot 2\cdot 3 \ldots \cdot n\cdot (n+1)} = \frac{1}{n+1}$

33. $\frac{n!}{(n+3)!} = \frac{1\cdot 2\cdot 3 \ldots \cdot n}{1\cdot 2\cdot 3 \ldots \cdot n\cdot (n+1)\cdot (n+2)\cdot (n+3)} = \frac{1}{(n+1)(n+2)(n+3)}$

35. $\frac{(n-1)!}{(n+1)!} = \frac{1\cdot 2\cdot 3 \ldots \cdot (n-1)}{1\cdot 2\cdot 3 \ldots \cdot (n-1)\cdot (n)\cdot (n+1)} = \frac{1}{(n)(n+1)}$

37. $\frac{(2n!)}{(2n+1)!} = \frac{1\cdot 2\cdot 3 \ldots \cdot n\cdot (n+1)\cdot \ldots \cdot (2n)}{1\cdot 2\cdot 3 \ldots \cdot n\cdot (n+1)\cdot \ldots \cdot (2n)\cdot (2n+1)} = \frac{1}{2n+1}$

39. $\frac{n^n}{n!} = \frac{n\cdot n\cdot n\cdot n\cdot \ldots \cdot n\cdot n}{1\cdot 2\cdot 3\cdot 4\cdot \ldots \cdot (n-1)\cdot n}$

41. Ratio = $\left| \frac{n}{n+1} \right| \to 1 = L$. Diverges (Harmonic series)

43. Ratio = $\left| \left(\frac{n}{n+1} \right)^3 \right| \to 1 = L$. Converges by the P–Test (p = 3).

45. Ratio = $\left| \frac{1}{2} \right| \to \frac{1}{2} = L$. Converges (Geometric series with r = 1/2).

47. Ratio = $|\, 1 \,| \to 1 = L$. Diverges by the N^{th} Term Test for Divergence.

49. Ratio = $\left| \frac{n!}{(n+1)!} \right| = \left| \frac{1}{n+1} \right| \to 0 = L$. Converges by the Ratio Test.

51. Ratio = $\left| \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} \right| = \left| \frac{2}{n+1} \right| \to 0 = L$. Converges by the Ratio Test.

53. Ratio = $\left| \frac{(1/2)^{3(n+1)}}{(1/2)^{3n}} \right| = \left| \left(\frac{1}{2}\right)^3 \right| = \frac{1}{8} \to \frac{1}{8} = L$. Converges by the Ratio Test.

55. Ratio = $\left| \frac{(0.9)^{2(n+1)+1}}{(0.9)^{2n+1}} \right| = \left| (0.9)^2 \right| = 0.81 \to 0.81 = L$. Converges by the Ratio Test.

57. Ratio = $\left| \frac{(x-5)^{n+1}}{(x-5)^n} \right| = |\, x-5 \,| \to |\, x-5 \,| = L$. Series converges absolutely if and only if $|\, x-5 \,| < 1$: $4 < x < 6$.

59. Ratio = $\left| \frac{(x-5)^{n+1}}{(n+1)^2} \frac{n^2}{(x-5)^n} \right| = \left| (x-5)\cdot \left(\frac{n}{n+1} \right)^2 \right| \to |\, x-5 \,| = L$. Series converges absolutely if and only if $|\, x-5 \,| < 1$: $4 < x < 6$.

61. Ratio = $\left| \frac{(x-2)^{n+1}}{(n+1)!} \frac{n!}{(x-2)n} \right| = \left| (x-2)\cdot \frac{1}{n+1} \right| \to 0 = L$ for all values of x so the series converges absolutely for all values of x.

63. Ratio = $\left| \frac{(2x-12)^{n+1}}{(n+1)^2} \frac{n^2}{(2x-12)^n} \right| = \left| (2x-12)\cdot \left(\frac{n}{n+1} \right)^2 \right| \to |\, 2x-12 \,| = L$. Series converges absolutely if and only if $|\, 2x-12 \,| < 1$: $\frac{11}{2} < x < \frac{13}{2}$.

65. Ratio = $\left| \frac{(6x-12)^{n+1}}{(n+1)!} \frac{n!}{(6x-12)^n} \right| = \left| (6x-12) \cdot \frac{1}{n+1} \right| \to 0 = L$ for all values of x so the series converges absolutely for all values of x.

67. Ratio = $\left| \frac{(x+1)^{2(n+1)}}{n+1} \frac{n}{(x+1)^{2n}} \right| = \left| (x+1)^2 \cdot \frac{n}{n+1} \right| \to (x+1)^2 = L$ for all values of x. Series converges absolutely if and only if $(x+1)^2 < 1$: $-2 < x < 0$.

69. Ratio = $\left| \frac{(x-5)^{3(n+1)+1}}{(n+1)^2} \frac{n^2}{(x-5)^{3n+1}} \right| = \left| (x-5)^3 \cdot \left(\frac{n}{n+1}\right)^2 \right| \to \left| (x-5)^3 \right| = L$ for all values of x. Series converges absolutely if and only if $\left| (x-5)^3 \right| < 1$: $4 < x < 6$.

71. Ratio = $\left| \frac{(x+3)^{2(n+1)-1}}{((n+1)+1)!} \frac{(n+1)!}{(x+3)^{2n-1}} \right| = \left| (x+3)^2 \cdot \frac{1}{n+2} \right| \to 0 = L$ for all values of x so the series converges absolutely for all values of x.

73. Ratio = $\left| \frac{x^{2(n+1)}}{(2(n+1))!} \frac{(2n)!}{x^{2n}} \right| = \left| x^2 \cdot \frac{1}{(2n+1)(2n+2)} \right| \to 0 = L$ for all values of x so the series converges absolutely for all values of x.

Section 10.7 Some Even Answers

2. Conditionally convergent
4. Conditionally convergent
6. Divergent
8. Absolutely convergent
10. Absolutely convergent
12. Conditionally convergent
14. Absolutely convergent
16. Conditionally convergent
18. Divergent
20. Absolutely convergent
22. Absolutely convergent
24. Absolutely convergent
26. Divergent
28. Divergent
30. Divergent
42. L = 1. Convergent by P–Test.
44. L = 1. Divergent by P–Test.
46. L = 1/3. Convergent Geo. series.
48. Divergent by N^{th} Term Test.
50. L=0. Convergent by Ratio Test.
52. L = 0. Convergent by Ratio Test.
54. $L = (1/3)^2$. Convergent by Ratio Test.
56. L = 0.64. Convergent by Ratio Test.
58. Absolutely convergent for $4 < x < 6$.
60. Absolutely convergent for $1 < x < 3$.
62. Absolutely convergent for all x.
64. Absolutely convergent for $11/4 < x < 13/4$.
66. Absolutely convergent for $2 < x < 4$.
68. Absolutely convergent for $-3 < x < -1$.
70. Absolutely convergent for all x.
72. Absolutely convergent for all x.
74. Absolutely convergent for all x.

Section 10.8 (Odd numbered problem solutions)

1. Ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{x^n} \right| = |x| \to |x| = L$. $|x| < 1$ if and only if $-1 < x < 1$.

   Endpoints: if $x = -1$ or $x = 1$, then the terms do not approach 0 so the series diverges.

   Interval of convergence: $-1 < x < 1$. (You can provide the graph of this interval.)

3. Ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x+2)^{n+1}}{(x+2)^n} \right| = |x+2| \to |x+2| = L$. $|x+2| < 1$ if and only if

   $-1 < x+2 < 1$ so $-3 < x < -1$.

   Endpoints: if $x = -3$ or $x = -1$, then the terms do not approach 0 so the series diverges.

   Interval of convergence: $-3 < x < -1$. (You can provide the graph of this interval.)

5. Ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}/(n+1)}{x^n/n} \right| = \left| x \cdot \frac{n}{n+1} \right| \to |x| = L$. $|x| < 1$ if and only if $-1 < x < 1$.

   Endpoints: if $x = -1$, then $\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alternating Series Test

   if $x = 1$, then $\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n}$ which diverges (harmonic series)

   Interval of convergence: $-1 \le x < 1$. (You can provide the graph of this interval.)

7. Ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x+3)^{n+1}/(n+1)}{(x+3)^n/n} \right| = \left| (x+3) \cdot \frac{n}{n+1} \right| \to |x+3| = L$. $|x+3| < 1$ if and only if $-1 < x+3 < 1$ or $-4 < x < -2$.

   Endpoints: if $x = -4$, then $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alternating Series Test

   if $x = -2$, then $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n}$ which diverges (harmonic series)

   Interval of convergence: $-4 \le x < -2$. (You can provide the graph of this interval.)

9. Ratio test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(x-7)^{2(n+1)+1}/(n+1)^2}{(x-7)^{2n+1}/n^2} \right| = \left| (x-7)^2 \cdot \left(\frac{n}{n+1} \right)^2 \right| \to (x-7)^2 = L$. $(x-7)^2 < 1$ if

   and only if $-1 < x-7 < 1$ or $6 < x < 8$.

   Endpoints: if $x = 6$, then $\sum_{n=1}^{\infty} \frac{(x-7)^{2n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{-1}{n^2}$ which converges by the P–Test

   if $x = 8$, then $\sum_{n=1}^{\infty} \frac{(x-7)^{2n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{(1)^{2n+1}}{n^2}$ which converges by the P–Test (p=2).

   Interval of convergence: $6 \le x \le 8$. (You can provide the graph of this interval.)

11. Ratio = $| 2x | \to | 2x | = L$. $| 2x | < 1$ if and only if $-1 < 2x < 1$ or $-1/2 < x < 1/2$.

Endpoints: if $x = -1/2$, then the series = $\sum_{n=1}^{\infty} (2\cdot(\frac{-1}{2}))^n = \sum_{n=1}^{\infty} (-1)^n$ which diverges

if $x = 1/2$, then the series = $\sum_{n=1}^{\infty} (2\cdot(\frac{1}{2}))^n = \sum_{n=1}^{\infty} (1)^n$ which diverges

Interval of convergence: $-1/2 < x < 1/2$. (You can provide the graph of this interval.)

13. Ratio = $\left| (\frac{x}{3})^2 \right| \to \left| \frac{x^2}{9} \right| = L$. $\left| \frac{x^2}{9} \right| < 1$ if and only if $-1 < \frac{x^2}{9} < 1$ or $-3 < x < 3$.

The series diverges at both endpoints, $x = -3$ and $x = 3$, so the interval of convergence is $-3 < x < 3$.

15. Ratio = $| 2x - 6 | \to | 2x - 6 | = L$. $| 2x - 6 | < 1$ if and only if $-1 < 2x - 6 < 1$ or $5/2 < x < 7/2$.

The series diverges at both endpoints, $x = 5/2$ and $x = 7/2$, so the interval of convergence is $5/2 < x < 7/2$.

17. Ratio = $\left| \frac{n!}{(n+1)!} \frac{x^{n+1}}{x^n} \right| = \left| \frac{1}{n+1} \cdot x \right| \to 0 = L$, and $L < 1$ for all x so the interval of convergence is the entire real number line.

19. Ratio = $\left| \frac{(n+1)^2}{n^2} \frac{3^n}{3^{n+1}} \frac{x^{n+1}}{x^n} \right| = \left| (\frac{n+1}{n})^2 \cdot \frac{x}{3} \right| \to \left| \frac{x}{3} \right| = L$. $\left| \frac{x}{3} \right| < 1$ if and only if $-1 < \frac{x}{3} < 1$.

The series diverges at both endpoints, $x = -3$ and $x = 3$, so the interval of convergence is $-3 < x < 3$.

21. Ratio = $\left| \frac{(n+1)!}{n!} \frac{x^{n+1}}{x^n} \right| = \left| (n+1)\cdot x \right| \to \infty > 1$ if $x \neq 0$ and $\left| (n+1)\cdot x \right| \to 0 < 1$ if $x = 0$.

The series diverges for $x \neq 0$, and the series converges (and is boring) when $x = 0$. The "interval" of convergence is a single point: $\{0\}$.

23. Ratio = $\left| \frac{(n+1)!}{n!} \frac{(x-7)^{n+1}}{(x-7)^n} \right| = \left| (n+1)\cdot(x-7) \right| \to \infty > 1$ if $x \neq 7$ and $\left| (n+1)\cdot(x-7) \right| \to 0 < 1$ if $x = 7$.

The series diverges for $x \neq 7$, and the series converges (and is boring) when $x = 7$. The "interval" of convergence is a single point: $\{7\}$.

25. Ratio = $\left| \frac{(x-a)^{n+1}}{(x-a)^n} \right| = \left| x - a \right| \to \left| x - a \right| = L$. $\left| x - a \right| < 1$ if and only if $a - 1 < x < a + 1$.

The series diverges at both endpoints, $x = a-1$ and $x = a+1$, so the interval of convergence is $a-1 < x < a+1$.

27. Ratio = $\left| \frac{(x-a)^{n+1}/(n+1)}{(x-a)^n/n} \right| = \left| \frac{n}{n+1} (x-a) \right| \to \left| x - a \right| = L$. $\left| x - a \right| < 1$ if and only if $a - 1 < x < a + 1$.

Endpoints: if $x = a-1$, then $\sum_{n=1}^{\infty} \frac{(x-a)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alternating Series Test

if $x = a+1$, then $\sum_{n=1}^{\infty} \frac{(x-a)^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n}$ which diverges (harmonic series)

Interval of convergence: $a-1 \le x < a+1$.

29. Ratio = $\left| \dfrac{(ax)^{n+1}}{(ax)^n} \right| = |ax| \to |ax| = L.$ $|ax| < 1$ if and only if $\dfrac{-1}{a} < x < \dfrac{1}{a}$.

The series diverges at both endpoints, $x = -1/a$ and $x = 1/a$, so the interval of convergence is $\dfrac{-1}{a} < x < \dfrac{1}{a}$.

31. Ratio = $\left| \dfrac{(ax-b)^{n+1}}{(ax-b)^n} \right| = |ax - b| \to |ax - b| = L.$ $|ax - b| < 1$ if and only if $-1 < ax - b \le 1$ or $\dfrac{b-1}{a} < x < \dfrac{b+1}{a}$.

The series diverges at both endpoints so the interval of convergence is $\dfrac{b-1}{a} < x < \dfrac{b+1}{a}$.

33. We can be certain the friend is wrong because the interval of convergence must be symmetric about the point $x = 4$, and the friend's interval, $1 < x < 9$, is not symmetric about $x = 4$.

35. (5, 9), [1, 13], (–1, 15], [3, 11), [0, 14), x = 7 are all possible intervals of convegence for the series.

37. (0, **2**), (**–5**, 7), [1, **1**], (**–3**, 5], [**–9**, 11], [0, **2**), x = **1**.

Note: There are many possible correct answers for problems 38 – 45.

39. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x}{3}\right)^n$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x-3}{3}\right)^n$

43. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x-5}{3}\right)^n$

45. $\displaystyle\sum_{n=1}^{\infty} n!\cdot(x-3)^n$

47. Interval of convergence: $2 < x < 4$. $\sum = \dfrac{1}{1-(x-3)} = \dfrac{1}{4-x}$.

49. Interval of convergence: $\dfrac{-1}{3} < x < \dfrac{1}{3}$. $\sum = \dfrac{1}{1-3x}$.

51. Interval of convergence: $-1 < x < 1$. $\sum = \dfrac{1}{1-x^3}$

53. Interval of convergence: $1 < x < 11$. $\sum = \dfrac{1}{1-\frac{x-6}{5}} = \dfrac{5}{11-x}$.

55. Interval of convergence: $-5 < x < 5$. $\sum = \dfrac{1}{1-\frac{x}{5}} = \dfrac{5}{5-x}$.

57. Interval of convergence: $-2 < x < 2$. $\sum = \dfrac{1}{1-(x/2)^3} = \dfrac{8}{8-x^3}$.

59. $\left|\frac{1}{3}\cos(x)\right| < 1$ **for all x**, so for all x the sum $\sum = \dfrac{1}{1-\frac{1}{3}\cos(x)} = \dfrac{3}{3-\cos(x)}$.

Section 10.9 (Odd numbered problem solutions)

1. $\frac{1}{1-x^4} = 1 + x^4 + x^8 + x^{12} + x^{16} + \ldots = \sum_{n=0}^{\infty} x^{4n}$

3. $\frac{1}{1+x^4} = 1 - x^4 + x^8 - x^{12} + x^{16} + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{4n}$

5. $\frac{1}{5+x} = \frac{1}{5} \cdot \frac{1}{1+(x/5)} = \frac{1}{5} \left\{ 1 - \frac{x}{5} + \left(\frac{x}{5}\right)^2 - \left(\frac{x}{5}\right)^3 + \left(\frac{x}{5}\right)^4 + \ldots \right\} = \frac{1}{5} \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{5}\right)^n$

7. $\frac{x^2}{1+x^3} = x^2 \left\{ \frac{1}{1+x^3} \right\} = x^2 \left\{ 1 - x^3 + x^6 - x^9 + x^{12} + \ldots \right\} = x^2 \sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n x^{3n+2}$

9. $\ln(1 + x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{n}$

11. $\arctan(x^2) = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \frac{x^{18}}{9} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2(2n+1)}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$

13. $\frac{1}{(1-x^2)^2} = 1 + 2x^2 + 3x^4 + 4x^6 + 5x^8 + \ldots = \sum_{n=1}^{\infty} n \cdot x^{2(n-1)} = \sum_{n=1}^{\infty} n \cdot x^{2n-2}$

15. $\int_0^{0.5} \frac{1}{1-x^3}\, dx \approx \int_0^{0.5} 1 + x^3 + x^6 + x^9 + \ldots\, dx = x + \frac{x^4}{4} + \frac{x^7}{7} + \ldots \Big|_0^{0.5} \approx 0.516741$

17. $\int_0^{0.6} \ln(1+x)\, dx \approx \int_0^{0.6} x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\, dx = \frac{x^2}{2} - \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} - \ldots \Big|_0^{0.6} = 0.1548$

19. $\int_0^{0.3} \frac{1}{(1-x)^2}\, dx \approx \int_0^{0.3} 1 + 2x + 3x^2 + \ldots\, dx = x + x^2 + x^3 + \ldots \Big|_0^{0.3} = 0.417$

21. $\lim_{x \to 0} \frac{\arctan(x)}{x} = \lim_{x \to 0} \frac{1}{x} \left\{ x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots \right\} = \lim_{x \to 0} 1 - \frac{x^2}{3} + \frac{x^4}{5} - \ldots = \mathbf{1}$

23. $\lim_{x \to 0} \frac{\ln(1+x)}{2x} = \lim_{x \to 0} \frac{1}{2x} \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \right\} = \lim_{x \to 0} \frac{1}{2} - \frac{x}{4} + \frac{x^2}{6} - \frac{x^3}{8} + \ldots = \mathbf{\frac{1}{2}}$

25. $\lim_{x \to 0} \frac{\ln(1-x^2)}{3x} = \lim_{x \to 0} \frac{1}{3x} \left\{ -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \frac{x^8}{4} - \ldots \right\} = \lim_{x \to 0} -\frac{x}{3} - \frac{x^3}{6} - \frac{x^5}{9} - \ldots = \mathbf{0}$

27. $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$. Using the Ratio Test, the ratio $= \left| \frac{x^{n+1}}{x^n} \right| = |x| \to |x| = L$. $|x| < 1$ if and only if $-1 < x < 1$.

The series diverges at the endpoints $x = -1$ and $x = 1$ so the interval of convergence is $-1 < x < 1$.

29. $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$. Using the Ratio Test, the ratio $= \left| \frac{x^{n+1}}{x^n} \frac{n}{n+1} \right| = \left| x \cdot \frac{n}{n+1} \right| \to |x| = L$.

$|x| < 1$ if and only if $-1 < x < 1$. The series converges when $x = -1$ and diverges when $x = 1$ so the interval of convergence is $-1 \le x < 1$.

31. $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

Using the Ratio Test, the ratio $= \left| \frac{x^{2(n+1)+1}}{x^{2n+1}} \frac{2n+1}{2(n+1)+1} \right| = \left| x^2 \cdot \frac{2n+1}{2n+3} \right| \to |x^2| = L$. $|x^2| < 1$ if and only if $-1 < x^2 < 1$ so $-1 < x < 1$. The series converges when $x = -1$ and when $x = 1$ so the interval of convergence is $-1 \le x \le 1$.

33. $\sin(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \ldots = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!}$$

35. $e^{(-x^2)} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \ldots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$

37. $\cos(x) = \mathbf{D}(\sin(x)) = \mathbf{D}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$

39. Using the result of Problem 33,

$$\int_0^1 \sin(x^2)\, dx = \int_0^1 x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \ldots\, dx = \left. \frac{x^3}{3} - \frac{x^7}{3! \cdot 7} + \frac{x^{11}}{5! \cdot 11} - \frac{x^{15}}{7! \cdot 15} + \ldots \right|_0^1$$

$$= \left\{ \frac{1}{3} - \frac{1}{3! \cdot 7} + \frac{1}{5! \cdot 11} - \frac{1}{7! \cdot 15} + \ldots \right\} - \{ 0 \}.$$

41. $\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}{x} = \lim_{x \to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots = 1 + (\text{all 0s}) = 1.$

Section 10.10 (Odd numbered problem solutions)

1. $P(x) = 4x^2 - 5x + 7$

2. $P(x) = 2x^3 - \frac{5}{2}x^2 + 2x - 1$

3. $P(x) = 2(x-3)^2 + 5(x-3) - 2$

5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

7. $\arctan(x) = x - \frac{x^3}{3} + \ldots$

9. $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

11. $\sec(x) = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \ldots$ (The higher derivatives of sec(x) get "messy.")

13. Around $c = 1$, $\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$

15. Around $c = \pi/2$, $\sin(x) = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \ldots$

17. Around $c = 9$, $\sqrt{x} = 3 + \frac{1}{2\cdot 3}(x-9) - \frac{1}{4\cdot 3^3}\frac{(x-9)^2}{2!} + \frac{3}{8\cdot 3^5}\frac{(x-9)^3}{3!} - \frac{15}{16\cdot 3^7}\frac{(x-9)^4}{4!} + \ldots$

19. Using the first three nonzero terms for cos(x),

$P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$. See the Cosine Table.

21. Using the first three nonzero terms for arctan(x),

$P(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$. See the Arctan Table.

x	cos(x)	P(x)
0.1	0.995004165	0.995004167
0.2	0.98006657	0.98006666
0.5	0.87758	0.87604
1	0.54030	0.54167
2	–0.4161	–0.3333

x	arctan(x)	P(x)
0.1	0.09966865	0.09966867
0.2	0.197396	0.197397
0.5	0.4636	0.4646
1	0.7854	0.8667
2	1.1071	5.7333

23. $\int \sin(x^2)\,dx = \int x^2 - \frac{x^6}{6} + \frac{x^{10}}{120}\,dx$

$= \frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} + C$

$\int \sin(x^3)\,dx = \int x^3 - \frac{x^9}{6} + \frac{x^{15}}{120}\,dx$

$= \frac{x^4}{4} - \frac{x^{10}}{60} + \frac{x^{16}}{1920} + C$

25. $\int e^{(-x^2)}\,dx = \int 1 + (-x^2) + \frac{(-x^2)^2}{2}\,dx = \int 1 - x^2 + \frac{x^4}{2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{10} + C$

$\int e^{(-x^3)}\,dx = \int 1 + (-x^3) + \frac{(-x^3)^2}{2}\,dx = \int 1 - x^3 + \frac{x^6}{2}\,dx = x - \frac{x^4}{4} + \frac{x^7}{14} + C$

27. $\int x\sin(x)\,dx = \int x\left\{ x - \frac{x^3}{6} + \frac{x^5}{120} \right\} dx = \int x^2 - \frac{x^4}{6} + \frac{x^6}{120}\,dx = \frac{x^3}{3} - \frac{x^5}{30} + \frac{x^7}{840} + C$

$\int x^2\sin(x)\,dx = \int x^2\left\{ x - \frac{x^3}{6} + \frac{x^5}{120} \right\} dx = \int x^3 - \frac{x^5}{6} + \frac{x^7}{120}\,dx = \frac{x^4}{4} - \frac{x^6}{36} + \frac{x^8}{960} + C$

29. $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{1 - \{ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \}}{x^2}$

$= \lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots}{x^2} = \lim_{x\to 0} \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \ldots = \mathbf{\frac{1}{2}}$.

31. $\lim_{x\to 0} \frac{1-e^x}{x} = \lim_{x\to 0} \frac{1 - \{ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \}}{x}$

$= \lim_{x\to 0} \frac{-x - \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots}{x} = \lim_{x\to 0} -1 - \frac{x}{2!} - \frac{x^2}{3!} - \ldots = \mathbf{-1}$

33. $\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots}{x} = \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots = 1$

35. $\lim_{x\to 0} \frac{x - \frac{x^3}{6} - \sin(x)}{x^5} = \lim_{x\to 0} \frac{x - \frac{x^3}{6} - \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots \right\}}{x^5}$

$= \lim_{x\to 0} \frac{-\frac{x^5}{5!} + \frac{x^7}{7!} \ldots}{x^5} = \lim_{x\to 0} -\frac{1}{5!} + \frac{x^2}{7!} \ldots = \mathbf{-\frac{1}{120}}$

37. $\sinh(x) = \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}\left(\{ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \} - \{ 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \} \right)$

$= \frac{1}{2}\left(2x + 2\frac{x^3}{3!} + 2\frac{x^5}{5!} + 2\frac{x^7}{7!} + \ldots \right) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$

39. $\mathbf{D}\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \right) = 1 + 3\frac{x^2}{3!} + 5\frac{x^4}{5!} + 7\frac{x^6}{7!} + \ldots = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

41. $e^{ix} = \cos(x) + i\cdot\sin(x)$. When $x = \pi/2$, then $e^{ix} = e^{i(\pi/2)} = \cos(\pi/2) + i\cdot\sin(\pi/2) = i$.

If $x = \pi$, then $e^{ix} = e^{\pi i} = \cos(\pi) + i\cdot\sin(\pi) = -1$. This result is often restated in the form

$e^{\pi i} + 1 = 0$, a formula that relates the 5 most common constants in all of mathematics, e, π, i, 1, and 0.

43. $\binom{3}{0} = 1$ (by the definition), $\binom{3}{1} = \frac{3!}{1!\bullet 2!} = \frac{3\bullet 2\bullet 1}{(1)\bullet(2\bullet 1)} = 3$,

$\binom{3}{2} = \frac{3!}{2!\bullet 1!} = \frac{3\bullet 2\bullet 1}{(2\bullet 1)\bullet(1)} = 3$, and $\binom{3}{3} = \frac{3!}{0!\bullet 3!} = \frac{3\bullet 2\bullet 1}{(1)\bullet(3\bullet 2\bullet 1)} = 1$.

45. The Maclaurin series for $(1 + x)^{5/2}$ is

$$1 + \frac{5}{2}x + (\frac{5}{2})(\frac{3}{2})\frac{x^2}{2!} + (\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\frac{x^3}{3!} + (\frac{5}{2})(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2})\frac{x^4}{4!} + (\frac{5}{2})(\frac{3}{2})(\frac{1}{2})(\frac{-1}{2})(\frac{-3}{2})\frac{x^5}{5!} + \ldots$$

47. If $f(x) = (1 + x)^m$, then $f(0) = 1$. $f'(x) = m(1 + x)^{m-1}$ so $f'(0) = m$.
$f''(x) = m(m-1)(1 + x)^{m-2}$ so $f''(0) = m(m-1)$. $f'''(x) = m(m-1)(m-2)(1 + x)^{m-3}$
so $f'''(0) = m(m-1)(m-2)$. $f^{(4)}(x) =$ m(m–1)(m–2)(m–3_$(1 + x)^{m-4}$ so $f'''(0) = m(m-1)(m-2)(m-3)$.
And so on. Then the Maclaurin series for $(1 + x)^m$ is

$$1 + mx + m(m-1)\frac{x^2}{2!} + m(m-1)(m-2)\frac{x^3}{3!} + m(m-1)(m-2)(m-3)\frac{x^4}{4!} + \ldots$$

Section 10.11 (Odd numbered problem solutions)

1. $f(x) = \sin(x)$, $c = 0$, $[-2, 4]$: $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f^{(iv)}(x) = \sin(x)$ so $a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -1/6, a_4 = 0$. Then $P_0(x) = 0$, $P_1(x) = 0 + x = x$, $P_2(x) = 0 + x + 0 = x$, $P_3(x) = 0 + x + 0 - x^3/6 = x - x^3/6$, and $P_4(x) = 0 + x + 0 - x^3/6 + 0 = x - x^3/6$.

3. $f(x) = \ln(x)$, $c = 1$, $[0.1, 3]$: $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 2/x^3$, $f^{(iv)}(x) = -6/x^4$ so (using $c = 1$) $a_0 = 0$, $a_1 = 1$, $a_2 = -1/2$, $a_3 = 1/3$, $a_4 = -1/4$. Then $P_0(x) = 0$, $P_1(x) = (x - 1)$, $P_2(x) = (x-1) - \frac{1}{2}(x-1)^2$, $P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$, and $P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$.

5. $f(x) = \sqrt{x} = x^{1/2}$, $c = 1$, $[1, 3]$: $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = \frac{-1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$, and $f^{(iv)}(x) = \frac{-15}{16}x^{-7/2}$ so $a_0 = 1$, $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{8}$, $a_3 = \frac{1}{16}$, $a_4 = \frac{-5}{128}$. Then $P_0(x) = 1$, $P_1(x) = 1 + \frac{1}{2}(x-1)$, $P_2(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2$,
$P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$, and
$P_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$.

7. $f(x) = (1+x)^{-1/2}$, $c = 0$, $[-2, 3]$: $f'(x) = \frac{-1}{2}(1+x)^{-3/2}$, $f''(x) = \frac{3}{4}(1+x)^{-5/2}$,
$f'''(x) = \frac{-15}{8}(1+x)^{-7/2}$, $f^{(iv)}(x) = \frac{105}{16}(1+x)^{-9/2}$ so $a_0 = 1$, $a_1 = \frac{-1}{2}$, $a_2 = \frac{3}{8}$, $a_3 = \frac{-5}{16}$,
and $a_4 = \frac{35}{128}$. Then $P_0(x) = 1$, $P_1(x) = 1 - \frac{1}{2}x$, $P_2(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2$,
$P_3(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3$, and $P_4(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4$.

9. $f(x) = \sin(x)$, $c = \pi/2$, $[-1, 5]$: $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f^{(iv)}(x) = \sin(x)$ so (using $c = \pi/2$) $a_0 = 1$, $a_1 = 0$, $a_2 = -1/2$, $a_3 = 0$, $a_4 = 1/4!$. Then $P_0(x) = 1$, $P_1(x) = 1$,
$P_2(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2$, $P_3(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2$, and $P_4(x) = 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 + \frac{1}{4!}(x - \frac{\pi}{2})^4$.

11. $f(x) = \sin(x)$, $c = 0$, $n = 5$, $-\pi/2 \le x \le \pi/2$: $R_5(x) = \frac{f^{(6)}(z)}{6!}(x-c)^6 = \frac{f^{(6)}(z)}{720}x^6$ for some z between x and 0.

$|R_5(x)| \le \frac{|-\sin(z)|}{720}|x|^6 \le \frac{1}{720}(\frac{\pi}{2})^6 \approx 0.02086$.

13. same as problem 11 except $-\pi \le x \le \pi$: $R_5(x) = \frac{f^{(6)}(z)}{6!}(x-c)^6 = \frac{f^{(6)}(z)}{720}x^6$ for some z between x and 0. $|R_5(x)| \le \frac{|-\sin(z)|}{720}|x|^6 \le \frac{1}{720}(\pi)^6 \approx 1.335$.

15. $f(x) = \cos(x)$, $c = 0$, $n = 10$, $-1 \le x \le 2$: $R_{10}(x) = \frac{f^{(11)}(z)}{11!}(x-c)^{11} = \frac{\sin(z)}{11!}x^{11}$ for some z between x and 0. $|R_{10}(x)| = \frac{|\sin(z)|}{11!}x^{11} \le \frac{1}{11!}(2)^{11} = \frac{2048}{39916800} \approx 0.000051$.

Since the only nonzero terms of the power series for $f(x) = \cos(x)$ (about $c = 0$) are the even powers of x, P_{10} is the same as P_{11} so we could use the error term for P_{11} instead of the one for P_{10}. The advantage of using the P_{11} error term is that we have a larger factorial in the denominator of R_{11} and get a smaller bound on the error of the approximation.

Taking $n = 11$, $-1 \le x \le 2$: $R_{11}(x) = \frac{f^{(12)}(z)}{12!}(x-c)^{12} = \frac{|\cos(z)|}{12!}x^{12}$ for some z between x and 0. $|R_{11}(x)| = \frac{|\cos(z)|}{12!}x^{12} \le \frac{1}{12!}(2)^{12} = \frac{4096}{479001600} \approx 0.00000855$.

17. $f(x) = e^x$, $c = 0$, $n = 6$, $-1 \le x \le 2$: $R_6(x) = \frac{f^{(7)}(z)}{7!}(x-c)^7 = \frac{f^{(7)}(z)}{5040}x^7$ for some z between x and 0. Using the estimate $e < 3$, $|R_6(x)| = \frac{|e^z|}{7!}|x|^7 < \frac{3^2}{7!}(2)^7 = \frac{1152}{5040} \approx 0.2286$.

19. $f(x) = \sin(x)$, $c = 0$, $[-1, 1]$, E = "error" = 0.001. The derivatives of $\sin(x)$ are $\pm\sin(x)$ or $\pm\cos(x)$ and all of them have maximum absolute value less than or equal to 1.

$| R_n(x) | = \frac{| f^{(n+1)}(z) |}{(n+1!)} |x - c|^{n+1} \le \frac{1}{(n+1)!} |1|^{n+1} = \frac{1}{(n+1)!}$ so we want to find a value of n that makes $\frac{1}{(n+1)!} \le 0.001$. It is difficult to solve factorial equations algebraically, but a "calculator investigation" shows that $\frac{1}{(5+1)!} = \frac{1}{720} \approx 0.00139 > 0.001$, and $\frac{1}{(6+1)!} = \frac{1}{5040} \approx 0.000198 < 0.001$ so we should **use n = 6**. That means we need to use the 3 terms involving x, x^3 and x^5.

21. $f(x) = \sin(x)$, $c = 0$, $[-1.6, 1.6]$, E = "error" = 0.00001. The derivatives of $\sin(x)$ are $\pm\sin(x)$ or $\pm\cos(x)$ and all of them have maximum absolute value less than or equal to 1.

$| R_n(x) | = \frac{| f^{(n+1)}(z) |}{(n+1!)} |x - c|^{n+1} \le \frac{1}{(n+1)!} |1.6|^{n+1} = \frac{1}{(n+1)!}(1.6)^{n+1}$ so we want to find a value of n that makes $\frac{1}{(n+1)!}(1.6)^{n+1} \le 0.00001$. Some calculator investigation shows that

$\frac{1}{(9+1)!}(1.6)^{9+1} \approx 0.00003 > 0.00001$ and

$\frac{1}{(10+1)!}(1.6)^{10+1} \approx 0.0000044 < 0.00001$ so we should **use n = 10**. That means we need to use the 5 terms involving x, x^3, x^5, x^7 and x^9.

23. $f(x) = e^x$, $c = 0$, $[0, 2]$, E = "error" = 0.001. The derivatives of e^x are all e^x .

$| R_n(x) | = \frac{| f^{(n+1)}(z) |}{(n+1!)} |x - c|^{n+1} \le \frac{| e^z |}{(n+1!)} |x|^{n+1} \le \frac{3^2}{(n+1)!} (2)^{n+1}$ (using $e < 3$).

Some calulator investigation shows $\frac{3^2}{(9+1)!} (2)^{9+1} \approx 0.0025 > 0.001$ and

$\frac{3^2}{(10+1)!} (2)^{10+1} \approx 0.00046 < 0.001$ so we should **use n = 10**. (Using a better upper bound for the value of e such as 2.75 or 2.72 does not change the conclusion: use n = 10.)

25. (a) $4\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \} \approx 4\{ 0.83492063 \} = 3.33968252$ compared with $\pi \approx 3.14159265$ from a calculator.

(b) Let $A = 4\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots - \frac{1}{99} \}$. Then $| A - \pi | < | 51^{st} \text{ term} | = \frac{4}{101} \approx 0.0396$.

(c) Let $B = 4\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \pm \frac{1}{2n-1} \}$.

Then $| B - \pi | < | \text{next term} | = \frac{4}{2(n+1) - 1} = \frac{4}{2n + 1}$. We want $\frac{4}{2n + 1} \le 0.0001$ so $n \ge 1999.5$. Take **n = 2000 terms** to get the precision we want.

27. (a) A: Put $C = 16\bullet\arctan(\frac{1}{5}) - 4\bullet\arctan(\frac{1}{239})$

$= 16\{\frac{1}{5} - (\frac{1}{3})(\frac{1}{5})^3 + (\frac{1}{5})(\frac{1}{5})^5\} - 4\{\frac{1}{239} - (\frac{1}{3})(\frac{1}{239})^3 + (\frac{1}{5})(\frac{1}{239})^5\}$

$\approx 16\{0.197397333333\} - 4\{0.004184076002\} = 3.14162102879$.

($C - \pi \approx 0.0000284$.)

(b) Formula A converges more rapidly than Methods I and II because we are using smaller values of x, x = 1/5 and x = 1/239, and the powers of these smaller values of x approach 0 more quickly than the values of x, x = 1 and x = 1/2 and x = 1/3, used in Methods I and II.