10.7 ABSOLUTE CONVERGENCE and the RATIO TEST

The series we examined in the previous sections all behaved very regularly with regard to the signs of the terms: the signs of the terms were either all the same or they alternated between + and – . However, there are series whose signs do not behave in such regular ways, and in this section we examine some techniques for determining whether those series converge or diverge.

Two Examples

Suppose we have the two series whose terms have magnitudes $\frac{1}{n}$ and $\frac{1}{n^2}$, but we don't know whether each term is positive or negative:

(a) $\sum_{n=1}^{\infty} * \frac{1}{n} = (*1) + (* \frac{1}{2}) + (* \frac{1}{3}) + (* \frac{1}{4}) + ... + (* \frac{1}{n}) + ...$ where each * is either + or – .

(b) $\sum_{n=1}^{\infty} * \frac{1}{n^2} = (*1) + (* \frac{1}{4}) + (* \frac{1}{9}) + (* \frac{1}{16}) + ... + (* \frac{1}{n^2}) + ...$ where each * is either + or – .

These two series behave very differently depending on how we replace the each "*" with + or – signs.

Series (a): $\sum_{n=1}^{\infty} * \frac{1}{n}$

If we **always** replace "*" with a +, then we have the harmonic series which diverges.

If we **always** replace "*" with a –, then we have the –1 times the harmonic series which diverges.

If we **alternate** replacing "*" with + and –, then we have the alternating harmonic series which converges.

The answer we have to give to the question "Does series (a) converge?" is "It depends on how the signs of the terms are chosen."

Series (b): $\sum_{n=1}^{\infty} * \frac{1}{n^2}$

If we always replace "*" with a +, then we have a series which converges (by the P–Test with p = 2). This is the largest value series (b) can have.

If we always replace "*" with a –, then we have –1 times a convergent series so the series converges. This is the smallest value series (b) can have.

If we replace the "*" with + or – in some other way, then the result is some number between the largest and smallest possible values (both of which are finite numbers), and the result must be some finite number.

The answer to the question "Does series (b) converge?" is "Yes, no matter how the * are replaced with + or – ." The value of the sum is affected by how the signs are chosen, but the series does converge.

Series (a) and (b) illustrate the distinction we want to examine in this section. Series (a) is an example of a **conditionally convergent** series since the convergence depends on how the "*" are replaced. Series (b) is an example of an **absolutely convergent** series since it does not matter how the "*" are replaced. Series (b) is convergent even when we are adding all positive terms or all negative terms.

The following definitions make the distinctions precise.

Definitions

Definition: A series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

Series (b) is absolutely convergent.

Definition: A series $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if it is convergent but not absolutely convergent (i.e., if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges).

The alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ is conditionally convergent because $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$ converges but $\sum_{n=1}^{\infty} \left| (-1)^{n+1}\frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example 1: Determine whether these series are absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$ (b) $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ (c) $\sum_{j=1}^{\infty} (-1)^{j+1}\frac{j^2}{j+1}$

Solution: (a) $\sum_{n=1}^{\infty} \left| (-1)^{n+1}\frac{1}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which diverges by the P–Test with $p = 1/2 < 1$ so the series is not absolutely convergent. $\frac{1}{\sqrt{n}} \longrightarrow 0$ monotonically so the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$ converges by the Alternating Series Test.

The series $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{\sqrt{n}}$ is conditionally convergent.

(b) $\sum_{k=1}^{\infty} \left| \frac{\sin(k)}{k^2} \right| \le \sum_{k=1}^{\infty} \frac{1}{k^2}$ which converges by the P–Test,

so $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ is absolutely convergent.

(c) $\left| (-1)^{j+1} \frac{j^2}{j+1} \right| = j - 1 + \frac{1}{j+1} \longrightarrow \infty \ne 0$ so $\sum_{j=1}^{\infty} (-1)^{j+1} \frac{j^2}{j+1}$ diverges.

Practice 1: Determine whether these series are absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=2}^{\infty} (-1)^n \frac{5}{\ln(n)}$ (b) $\sum_{k=1}^{\infty} \frac{\cos(\pi k)}{k^2}$

An important result about absolutely convergent series is that they are also convergent.

Absolute Convergence Theorem

Every absolutely convergent series is convergent: if $\sum_{n=1}^{\infty} | a_n |$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Since the series $\sum_{n=1}^{\infty} \left| * \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent, the series $\sum_{n=1}^{\infty} * \frac{1}{n^2}$ converges no matter how the "*" is replaced with + and – signs.

Proof of the Absolute Convergence Theorem:

If $a_n \ge 0$ then $a_n = | a_n |$, and if $a_n < 0$ then $a_n = -| a_n |$. In either case we have $-| a_n | \le a_n \le | a_n |$.

Adding $| a_n |$ to each piece of $-| a_n | \le a_n \le | a_n |$, we have $0 \le | a_n | + a_n \le 2 | a_n |$ for all n.

Let $b_n = | a_n | + a_n \ge 0$ for all n.

Since $b_n \ge 0$ and $b_n \le 2 | a_n |$, the terms of a convergent series, then, by the Comparison Test we know that $\sum_{n=1}^{\infty} b_n$ converges.

Finally, $a_n = (| a_n | + a_n) - | a_n | = b_n - | a_n |$ so the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} | a_n |$

is the difference of two convergent series, and we can conclude that the series $\sum_{n=1}^{\infty} a_n$ converges.

The following corollary (the contrapositive form of the Absolute Convergence Theorem) is sometimes useful for showing that a series is not absolutely convergent.

Corollary:

If a series is not convergent, then it is not absolutely convergent:

if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

The Ratio Test

The following test is useful for determining whether a given series is absolutely convergent, and it will be used often in Section 10.8 when we want to determine where a power series converges. It says that we can be certain that a given series is absolutely convergent, if the limit of the ratios of successive terms has a value less than 1. The Ratio Test is very important and will be used very often in the next sections on power series.

The Ratio Test

Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

(a) If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and also convergent).

(b) If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $L = 1$, then the series $\sum_{n=1}^{\infty} a_n$ may converge or may diverge (the Ratio Test does not help).

A proof of the Ratio Test is rather long and is included in an Appendix after the Practice Answers for this section. Part (a) is proved by showing that if $L < 1$, then the series is, term–by–term, less than a convergent geometric series. Part (b) is proved by showing that if $L > 1$, then the terms of the series do not approach 0 so the series diverges. Part (c) is proved by giving two series, one convergent and one divergent, that both have $L = 1$.

One powerful aspect of the Ratio Test is that it is very "mechanical" — we simply calculate a particular limit and then we (often) have a conclusion about the convergence or divergence of the series.

Example 2: Use the Ratio Test to determine if these series are absolutely convergent:

(a) $\sum_{n=1}^{\infty} \frac{2^n n}{5^n}$ (b) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

Solution: (a) $a_n = \frac{2^n n}{5^n}$ so $a_{n+1} = \frac{2^{n+1}(n+1)}{5^{n+1}}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{2^{n+1}(n+1)}{5^{n+1}}}{\frac{2^n n}{5^n}} \right| = \left| \frac{2^{n+1}}{2^n} \frac{5^n}{5^{n+1}} \frac{n+1}{n} \right| = \left| \frac{2}{5} \frac{n+1}{n} \right| \longrightarrow \frac{2}{5} < 1$$

so $\sum_{n=1}^{\infty} \frac{2^n n}{5^n}$ is absolutely convergent.

(b) $a_n = \frac{n^2}{n!}$ so $a_{n+1} = \frac{(n+1)^2}{(n+1)!}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} \right| = \left| \frac{n!}{(n+1)!} \frac{(n+1)^2}{n^2} \right| = \left| \frac{1}{n+1} \left(\frac{n+1}{n}\right)^2 \right| \longrightarrow 0 < 1$$

so $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ is absolutely convergent.

Practice 2: Use the Ratio Test to determine if these series are absolutely convergent:

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^n}{n!}$ (b) $\sum_{n=1}^{\infty} \frac{n^5}{3^n}$.

The Ratio Test is very useful for determining values of a variable which guarantee the absolute convergence (and convergence) of a series. This is a method we will use often in the rest of this chapter.

Example 3: For which values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ absolutely convergent?

Solution: $a_n = \frac{(x-3)^n}{n}$ so $a_{n+1} = \frac{(x-3)^{n+1}}{n+1}$. Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right| = \left| \frac{(x-3)^{n+1}}{(x-3)^n} \frac{n}{n+1} \right| \rightarrow |x-3| = L.$$

Now we simply need to solve the inequality $|x-3| < 1$ ($L < 1$) for x.

If $|x-3| < 1$, then $-1 < x-3 < 1$ so $\mathbf{2 < x < 4}$.

If $2 < x < 4$, then $L = |x-3| < 1$ so $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ is absolutely convergent (and convergent).

If $x < 2$ or $x > 4$ then $L > 1$ so the series diverges. (What happens at the "endpoints," x=2 and x=4, where L=1?)

Practice 3: For which values of x is the series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$ absolutely convergent?

Note: If the terms of a series contain **factorials** or things raised to the **nth power**, it is usually a good idea to use the Ratio Test as the first test you try.

Rearrangements

Absolutely convergent series share an important property with finite sums — no matter what order we add the numbers, the sum is always the same. Stated another way, the sum of an absolutely convergent series is always the same value, even if the terms are rearranged.

Conditionally convergent series do not have this property — the order in which we add the terms does matter. If the terms of a conditionally convergent series are rearranged or reordered, the sum after the rearrangement may be different than the sum before the rearrangement. A rather amazing fact is that the terms of a conditionally convergent series can be rearranged to obtain any sum we want.

We illustrate this strange result by showing that we can rearrange the alternating harmonic series, a series conditionally convergent to approximately 0.69, so that the sum of the rearranged series is 2 rather than 0.69..

$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$ is conditionally convergent to approximately 0.69 .

First we note that the sum of the positive terms alone is a divergent series:

$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots = \sum_{n=1}^{\infty} \frac{1}{2n-1}$ which diverges by the Limit Comparison Test,

so the partial sums of the positive terms eventually exceed any number we pick.

Similarly, the sum of the negative terms alone is also a divergent series:

$-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \ldots = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges,

so the partial sums of the negative terms eventually become as large negatively as we want.

Finally, we pick a target number we want for the sum (in this illustration, we picked a target of 2).

Then the following clever strategy tells us how to chose the order of the terms, the rearrangement, so the sum of the rearranged series is the target number, 2 :

(1) select the positive terms, in the order they appear in the original series, until the partial sum exceeds our target number, then

(2) select the negative terms, in the order they appear in the original series, until the partial sum falls below our target number, and

(3) keep repeating steps (1) and (2) with the previously unused terms of the original series.

In order to rearrange the terms of the alternating harmonic series so the sum is 2, we pick positive terms, in order, until the partial sum **is larger than** 2. This requires the first 8 positive terms:

$s_1 = 1,$

$s_2 = 1 + \frac{1}{3} \approx 1.3333, \ldots$

$s_7 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} \approx 1.955133755$

$s_8 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} \approx 2.021800422 .$

Then we pick negative terms, in order, until the partial sum **is less than** two. This only requires 1 negative term:

$s_9 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} - \mathbf{\frac{1}{2}} \approx 1.521800422 .$

Then we pick more unused positive terms until the partial sum exceeds 2. This requires many more positive terms:

$s_{10} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} - \frac{1}{2} + \frac{1}{17} \approx 1.580623951$

$s_{11} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} - \frac{1}{2} + \frac{1}{17} + \frac{1}{19} \approx 1.63325553$

$s_{12} = s_{11} + \frac{1}{21} \approx 1.680874578$, $s_{13} = s_{12} + \frac{1}{23} \approx 1.724352839$

$s_{14} \approx 1.764352839$, $s_{15} \approx 1.801389876$, $s_{16} \approx 1.835872634$, $s_{17} \approx 1.868130699$

$s_{18} \approx 1.898433729$, $s_{19} \approx 1.927005158$, $s_{20} \approx 1.954032185$, $s_{21} \approx 1.97967321$,

$s_{22} \approx s_{21} + \frac{1}{41} \approx 2.004063454 .$

Then we pick more previously unused negative terms, in order, until the partial sum is less than two. Again only 1 negative term is required:

$s_{23} = s_{22} - \mathbf{\frac{1}{4}} \approx 1.754063454 .$

As we continue to repeat this process, we "eventually" use all of the terms of the original conditionally convergent series and the partial sums of the new "rearranged" series get, and stay, arbitrarily close to the target number, 2. The same method can be used to rearrange the terms of the alternating harmonic series (or any conditionally convergent series) to sum to 0.3, 3, 30 or any positive target number we want. How do you think the strategy needs to be changed to rearrange a conditionally convergent series to sum to a **negative** target number?

PROBLEMS

In problems 1 – 30 determine whether the given series Converge Absolutely, Converge Conditionally, or Diverge and give reasons for your conclusions.

1. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+2}$
2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$
3. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5}{n^3}$
4. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{2 + \ln(n)}$
5. $\sum_{n=1}^{\infty} (-0.5)^n$
6. $\sum_{n=1}^{\infty} (-0.5)^{-n}$

7. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$

8. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{3+n^2}$

9. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n)}{n}$

10. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln(n)}{n^2}$

11. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+\ln(n)}$

12. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{5}{\sqrt{n+7}}$

13. $\sum_{n=1}^{\infty} (-1)^n \sin(\frac{1}{n})$ (recall that if $0 < x < 1$, then $0 < \sin(x) < x$)

14. $\sum_{n=1}^{\infty} (-1)^n \sin(\frac{1}{n^2})$

15. $\sum_{n=1}^{\infty} (-1)^n \sqrt{n} \sin(\frac{1}{n^2})$

16. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$

17. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln(n)}{\ln(n^3)}$

18. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+7}{n^2+10}$

19. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+7}{n^3+10}$

20. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+7}{n^4+10}$

21. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n^2+7)^2}{n^2+10}$

22. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

23. $\sum_{n=1}^{\infty} \frac{\sin(\pi n)}{n}$

24. $\sum_{n=1}^{\infty} (-n)^{-n}$

25. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+\sqrt{3n}}{n+2}$

26. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

27. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-3)^n}{n^3}$

28. $\sum_{n=2}^{\infty} (-1)^{n+1} \left(\frac{\ln(n)}{\ln(n^5)}\right)^2$

29. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n \cdot 3^n}$

30. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

The Ratio Test is commonly used with series that contain factorials, and factorials are also going to become more common in the next few sections. Problems 31 – 40 ask you to simplify factorial expressions in order to get ready for that usage.

(Definitions: $0! = 1$, $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, $4! = 24$, and, in general, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$)

31. Show that $\frac{n!}{(n+1)!} = \frac{1}{n+1}$

32. Show that $\frac{n!}{(n+2)!} = \frac{1}{(n+1)(n+2)}$

33. Show that $\frac{n!}{(n+3)!} = \frac{1}{(n+1)(n+2)(n+3)}$

34. Show that $\frac{(n+1)!}{(n+2)!} = \frac{1}{n+2}$

35. Show that $\frac{(n-1)!}{(n+1)!} = \frac{1}{(n)(n+1)}$

36. Show that $\frac{2(n!)}{(2n)!} = \frac{2}{(n+1) \cdot (n+2) \cdot \ldots \cdot (2n)}$

37. Show that $\frac{(2n)!}{(2n+1)!} = \frac{1}{2n+1}$

38. Show that $\frac{(2n)!}{(2(n+1))!} = \frac{1}{(2n+1)(2n+2)}$

39. Show that $\frac{n^n}{n!} = \frac{n}{1} \cdot \frac{n}{2} \cdot \frac{n}{3} \cdot \ldots \cdot \frac{n}{n-1} \cdot \frac{n}{n}$

40. For $n > 0$, which is larger, $7!$ or $\frac{(n+7)!}{n!}$?

In problems 41 – 56, (a) determine the value of $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ for the given series, and (b) state the conclusion of the Ratio Test as it applies to the series. (c) If the Ratio Test is inconclusive, use some other method to determine if the given series converges or diverges.

41. $\sum_{n=1}^{\infty} \frac{1}{n}$

42. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

43. $\sum_{n=2}^{\infty} \frac{1}{n^3}$

44. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$

45. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

46. $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

47. $\sum_{n=5}^{\infty} 1^n$

48. $\sum_{n=1}^{\infty} (-2)^n$

49. $\sum_{n=1}^{\infty} \frac{1}{n!}$

50. $\sum_{n=1}^{\infty} \frac{5}{n!}$

51. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

52. $\sum_{n=1}^{\infty} \frac{5^n}{n!}$

53. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{3n}$

54. $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{2n}$

55. $\sum_{n=5}^{\infty} (0.9)^{2n+1}$

56. $\sum_{n=5}^{\infty} (-0.8)^{2n+1}$

For each series in problems 57 – 74 find the values of x for which the value of $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$. From the Ratio Test we can conclude that each series converges absolutely (and thus converges) for those values of x.

57. $\sum_{n=5}^{\infty} (x-5)^n$

58. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$

59. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$

60. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$

61. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n!}$

62. $\sum_{n=1}^{\infty} \frac{(x-10)^n}{n!}$

63. $\sum_{n=1}^{\infty} \frac{(2x-12)^n}{n^2}$

64. $\sum_{n=1}^{\infty} \frac{(4x-12)^n}{n^2}$

65. $\sum_{n=1}^{\infty} \frac{(6x-12)^n}{n!}$

66. $\sum_{n=1}^{\infty} (x-3)^{2n}$

67. $\sum_{n=2}^{\infty} \frac{(x+1)^{2n}}{n}$

68. $\sum_{n=2}^{\infty} \frac{(x+2)^{2n+1}}{n^2}$

69. $\sum_{n=1}^{\infty} \frac{(x-5)^{3n+1}}{n^2}$

70. $\sum_{n=1}^{\infty} \frac{(x+4)^{2n+1}}{n!}$

71. $\sum_{n=1}^{\infty} \frac{(x+3)^{2n-1}}{(n+1)!}$

72. $S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

73. $C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

74. $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Rearrangements

In problems 75 – 80 use the strategy in the illustrative example to find the first 15 terms of a rearrangement of the given conditionally convergent series so that the rearranged series converges to the given target number.

75. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, target number = 0.3 .

76. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$, target number = 0.7 .

77. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$, target number = 1 .

78. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$, target number = 0.4 .

79. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$, target number = 1 .

80. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$, target number = –1 .

Practice Answers

Practice 1: (a) $\sum_{n=2}^{\infty} (-1)^n \frac{5}{\ln(n)}$ converges by the Alternating Series Test

$\sum_{n=2}^{\infty} \left| (-1)^n \frac{5}{\ln(n)} \right| = \sum_{n=2}^{\infty} \frac{5}{\ln(n)}$ which diverges by comparison with the divergent series $\sum_{n=2}^{\infty} \frac{5}{n}$.

Therefore, $\sum_{n=2}^{\infty} (-1)^n \frac{5}{\ln(n)}$ is **conditionally convergent** .

(b) $\sum_{k=1}^{\infty} \frac{\cos(\pi k)}{k^2} = \frac{-1}{1^2} + \frac{1}{2^2} + \frac{-1}{3^2} + \ldots$

$\sum_{k=1}^{\infty} \left| \frac{\cos(\pi k)}{k^2} \right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ which is convergent (P–test, p = 2) so

$\sum_{k=1}^{\infty} \frac{\cos(\pi k)}{k^2}$ is **absolutely convergent** (and convergent).

Practice 2: (a) $a_n = (-1)^{n+1}\frac{e^n}{n!}$ so $a_{n+1} = (-1)^{n+2}\frac{e^{n+1}}{(n+1)!}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2}\frac{e^{n+1}}{(n+1)!}}{(-1)^{n+1}\frac{e^n}{n!}}\right| = \left|\frac{e^{n+1}}{e^n}\,\frac{n!}{(n+1)!}\right| = \left|\frac{e}{n+1}\right| \longrightarrow 0 < 1$$

so $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^n}{n!}$ is absolutely convergent.

(b) $a_n = \frac{n^5}{3^n}$ so $a_{n+1} = \frac{(n+1)^5}{3^{n+1}}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)^5}{3^{n+1}}}{\frac{n^5}{3^n}}\right| = \left|\frac{3^n}{3^{n+1}}\,\frac{(n+1)^5}{n^5}\right| = \left|\frac{1}{3}\left(\frac{n+1}{n}\right)^5\right| \longrightarrow \frac{1}{3} < 1$$

so $\sum_{n=1}^{\infty}\frac{n^5}{3^n}$ is absolutely convergent.

Practice 3: $a_n = \frac{(x-5)^n}{n^2}$ so $a_{n+1} = \frac{(x-5)^{n+1}}{(n+1)^2}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(x-5)^{n+1}}{(n+1)^2}}{\frac{(x-5)^n}{n^2}}\right| = \left|\frac{(x-5)^{n+1}}{(x-5)^n}\,\frac{n^2}{(n+1)^2}\right| \rightarrow |x-5| = L.$$

Now we simply need to solve the inequality $|x-5| < 1$ ($L < 1$) for x.

If $|x-5| < 1$, then $-1 < x-5 < 1$ so $\mathbf{4 < x < 6}$.

If $4 < x < 6$, then $L = |x-5| < 1$ so $\sum_{n=1}^{\infty}\frac{(x-5)^n}{n^2}$ is absolutely convergent (and convergent).

Also, if $x < 4$ or $x > 6$ then $L > 1$ so $\sum_{n=1}^{\infty}\frac{(x-5)^n}{n^2}$ diverges.

"Endpoints": If $x = 4$, then $\sum_{n=1}^{\infty}\frac{(x-5)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ which converges.

If $x=6$, then $\sum_{n=1}^{\infty}\frac{(x-5)^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ which converges.

Appendix: A Proof of the Ratio Test

The Ratio Test has three parts, (a), (b), and (c), and each part requires a separate proof.

(a) $L<1 \Rightarrow$ the series is absolutely convergent. The basic pattern of the proof for this part is to show that the given series is, term–by–term, less than a convergent geometric series. Then we conclude by the Comparison Test that the given series converges.

Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$. Then there is a number r between L and 1 so that the ratios $\left| \frac{a_{n+1}}{a_n} \right|$ are eventually (for all $n >$ some N) less than r :

for all $n > N$, $\left| \frac{a_{n+1}}{a_n} \right| < r < 1$.

Then $| a_{N+1} | < r | a_N |$, $| a_{N+2} | < r | a_{N+1} | < r^2 | a_N |$, $| a_{N+3} | < r | a_{N+2} | < r^3 | a_N |$,
and, in general, $| a_{N+k} | < r^k | a_N |$.

So $|a_N| + |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + |a_{N+4}| + \ldots$

$< | a_N | + r | a_N | + r^2 | a_N | + r^3 | a_N | + r^4 | a_N | + \ldots$

$= | a_N |\cdot\{ 1 + r + r^2 + r^3 + r^4 + \ldots \}$

$= | a_N |\cdot \frac{1}{1-r}$, since the powers of r form a convergent geometric series,

and the series $\sum_{n=N}^{\infty} | a_n |$ is convergent by the Comparison Test.

Finally, we can conclude that the series $\sum_{n=1}^{\infty} | a_n | = \sum_{n=1}^{N-1} | a_n | + \sum_{n=N}^{\infty} | a_n |$ is convergent, since it is the sum of two convergent series. $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) $L>1 \Rightarrow$ the series is divergent. The basic idea in this part is to show that the terms of the given series do not approach 0. Then we can conclude by the N^{th} Term Test that the given series diverges.

Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$. Then the ratios $\left| \frac{a_{n+1}}{a_n} \right|$ are eventually (for all $n >$ some N) larger than 1 so $| a_{N+1} | > | a_N |$, $| a_{N+2} | > | a_{N+1} | > | a_N |$, $| a_{N+3} | > | a_{N+2} | > | a_N |$,
and, for all $k > N$, $| a_k | > | a_N |$.

Thus $\lim_{n\to\infty} | a_n | \ge | a_N | \ne 0$ and $\lim_{n\to\infty} a_n \ne 0$, so by the N^{th} Term Test for Divergence (Section 10.2) we can conclude that the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) L=1 ⇒ nothing. Part (c) can be verified by giving two series, one convergent and one divergent, for which L = 1.

If $a_n = \frac{1}{n^2}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by the P–Test with p=2, and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{1/(n+1)^2}{1/n^2} \right| = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2 = 1.$$

If $a_n = \frac{1}{n}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series, and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{1/(n+1)}{1/n} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since L =1 for each of these series, one convergent and one divergent, knowing that L = 1 for a series does not let us conclude that the series converges or that it diverges.