10.4 POSITIVE TERM SERIES: INTEGRAL TEST & P–TEST

This section discusses two methods for determining whether some series are convergent. The first, the integral test, says that a given series converges if and only if a related improper integral converges. This lets us trade a question about the convergence of a series for a question about the convergence of an improper integral. The second convergence test, the P–test, says that the convergence of one particular type of series, the sum $\sum_{k=1}^{\infty} \frac{1}{k^p}$, depends only on the value of p. These tests only apply to series whose terms are positive. And, unfortunately, the tests only tell us if the series converge or diverge, but they do not tell us the actual sum of the series.

The Integral Test is the more fundamental and general of the two tests examined in this section, and it is used to prove the P–Test. The P–Test, however, is easier to apply and is likely to be the test you use more often.

Integral Test

A series can be thought of as a sum of areas of rectangles each having a base of one unit (Fig. 1). With this area interpretation of series there is a natural connection between series and integrals and between the convergence of a series and the convergence of an appropriate improper integral.

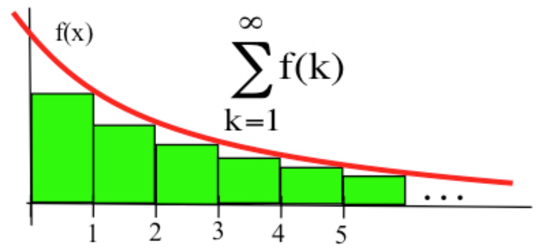

Fig. 1

Example 1: Suppose the shaded region in Fig. 2a can be painted using 3 gallons of paint. How much paint is needed for the shaded region in Fig. 2b?

Solution: We don't have enough information to determine the exact amount of paint needed for the region in Fig. 2b, but the total of the rectangular areas is smaller than the area in Fig. 2a so less than 3 gallons of paint are needed for the region in Fig. 2b.

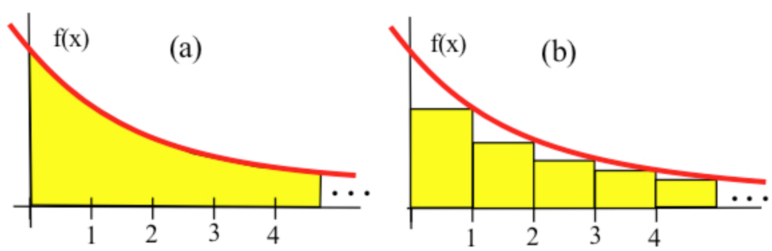

Fig. 2

Practice 1: Suppose the area of the shaded region in Fig. 3a is infinite. What can you say about the total area of the rectangular regions in Fig. 3b?

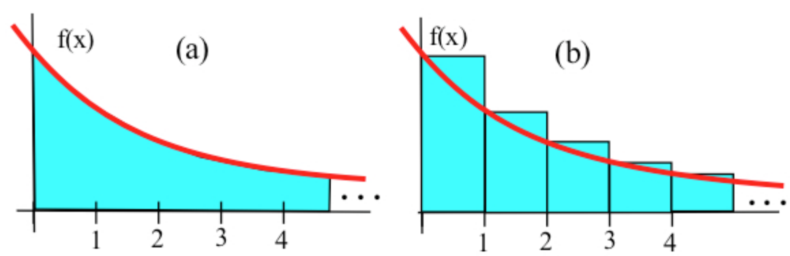

Fig. 3

The geometric reasoning used in Example 1 and Practice 1 can also be used to determine the convergence and divergence of some series.

Example 2: (a) Which is larger: $\sum_{k=2}^{\infty} \frac{1}{k^2}$ or $\int_1^{\infty} \frac{1}{x^2}\, dx$?

(b) Use the result of (a) to show that $\sum_{k=2}^{\infty} \frac{1}{k^2}$ is convergent.

Solution: Fig. 4 illustrates that the area of the rectangles, $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... + \frac{1}{n^2}$, is less than the area under the graph of the function $f(x) = \frac{1}{x^2}$ for $1 \le x \le n$:

$$\sum_{k=2}^{n} \frac{1}{k^2} < \int_1^{n} \frac{1}{x^2}\, dx \quad \text{so} \quad \sum_{k=2}^{n} \frac{1}{k^2} < \int_1^{\infty} \frac{1}{x^2}\, dx = 1 \quad \text{for every } n \ge 2.$$

Therefore, the partial sums of $\sum_{k=2}^{\infty} \frac{1}{k^2}$ are bounded. Also, each term $a_k = \frac{1}{k^2}$ is positive, so the partial sums of $\sum_{k=2}^{\infty} \frac{1}{k^2}$ are monotonically increasing. So, by the Monotonic Theorem of Section 10.1, the sequence of partial sums converges, so the series $\sum_{k=2}^{\infty} \frac{1}{k^2}$ is convergent.

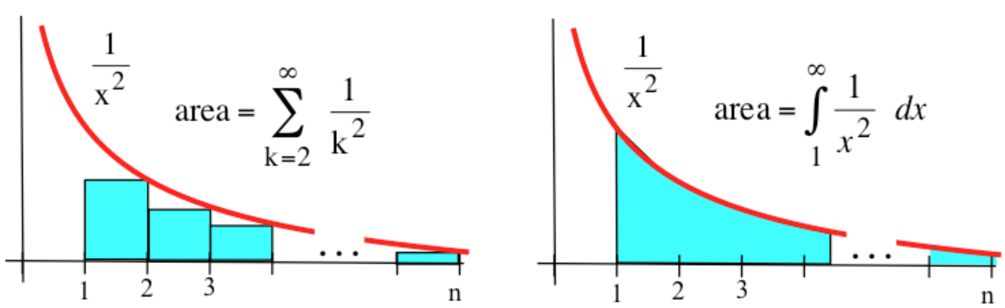

Fig. 4

The reasoning of Example 2 can be extended to the comparison of other series and the appropriate integrals.

Integral Test

Suppose f is a **continuous, positive, decreasing** function on $[1, \infty)$ and $a_k = f(k)$.

The series $\sum_{k=1}^{\infty} a_k$ converges **if and only if** the integral $\int_1^{\infty} f(x)\,dx$ converges.

Equivalently, (a) if $\int_1^{\infty} f(x)\,dx$ converges, then $\sum_{k=1}^{\infty} a_k$ converges,

and (b) if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

The proof is simply a careful use of the reasoning in the previous Examples.

Proof: Assume that f is a continuous, positive, decreasing function on $[1, \infty)$ and that $a_k = f(k)$.

Part (a): Assume that $\int_1^{\infty} f(x)\,dx$ converges: $\lim_{n\to\infty} \int_1^{n} f(x)\,dx$ is a finite number .

Since each $a_k > 0$, the sequence of partial sums s_n is increasing. If we arrange the rectangles under the graph of f as in Fig. 5, it is clear that

$$s_n = \sum_{k=1}^{n} a_k = a_1 + \sum_{k=2}^{n} a_k \le a_1 + \int_1^{n} f(x)\,dx \le a_1 + \int_1^{\infty} f(x)\,dx .$$

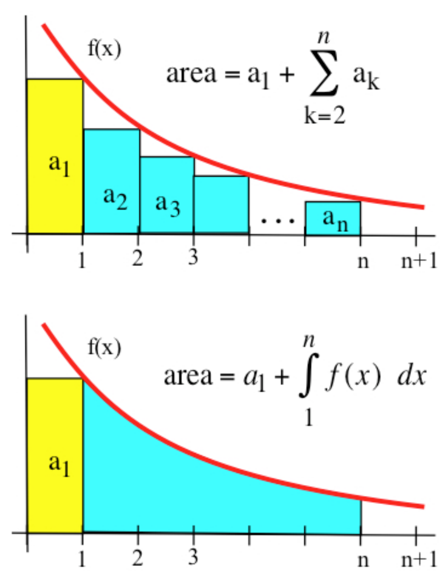

Fig. 5

$\{ s_n \}$ is a bounded, increasing sequence so, by the Monotone Convergence Theorem of Section 10.1, $\{ s_n \}$ converges and $\sum_{k=1}^{\infty} a_k$ is convergent.

Part (b): Assume that $\int_1^\infty f(x)\,dx$ diverges: $\lim_{n\to\infty} \int_1^n f(x)\,dx = \infty$.

If we arrange the rectangles under the graph of f as in Fig. 6, it is clear that

$$s_n = \sum_{k=1}^{n} a_k \ \ge \int_1^{n+1} f(x)\,dx \quad \text{for all } n,$$

$$\text{so} \quad \lim_{n\to\infty} s_n \ge \lim_{n\to\infty} \int_1^n f(x)\,dx = \infty .$$

In other words, $\lim_{n\to\infty} s_n = \infty$ and $\sum_{k=1}^{\infty} a_k$ diverges.

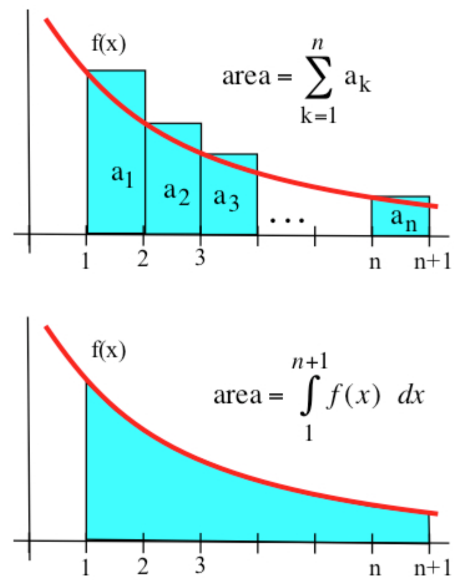

Fig. 6

The inequalities in the proof relating the partial sums of the series to the values of integrals are sometimes used to approximate the values of the partial sums of a series:

$$\int_1^{n+1} f(x)\,dx \ \le \ \sum_{k=1}^{n} a_k \ \le \ a_1 + \int_1^{n} f(x)\,dx \ .$$

We can use this last set of inequalities with $a_k = \frac{1}{k}$ and $n = 1{,}000$ to conclude that

$$\int_1^{10^3+1} \frac{1}{x}\,dx \ \le \ \sum_{k=1}^{10^3} \frac{1}{k} \ \le \ 1 + \int_1^{10^3} \frac{1}{x}\,dx \quad \text{so} \quad 7.48646986155 \le \sum_{k=1}^{10^3} \frac{1}{k} \ \le \ 8.48547086055 \ .$$

(If $n = 1{,}000{,}000$, then the same type of reasoning shows that the partial sum of $1/k$ from $k = 1$ to $k = 10^6$ is between 13.815511 and 14.815510 .)

If the series does not start with $k = 1$, a Corollary of the Integral Test can be used.

Corollary: If f satisfies the hypotheses of the Integral Test on $[N, \infty)$ and $a_k = f(k)$,

then $\sum_{k=N}^{\infty} a_k$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Example 3: Use the Integral Test to determine whether (a) $\sum_{k=1}^{\infty} \frac{1}{k^3}$ and (b) $\sum_{k=2}^{\infty} \frac{1}{k \cdot \ln(k)}$ converge.

Solution: (a) If $f(x) = \frac{1}{x^3}$, then $a_k = f(k)$ and f is continuous, positive and decreasing on $[1, \infty)$.

$$\text{Then } \int_1^\infty \frac{1}{x^3}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x^3}\,dx = \lim_{n\to\infty}\left(-\frac{1}{2x^2}\right)\Bigg|_1^n$$

$$= \lim_{n\to\infty}\left(-\frac{1}{2n^2}\right) - \left(-\frac{1}{2\cdot 1^2}\right) = \frac{1}{2}.$$

The integral $\int_1^\infty \frac{1}{x^3}\,dx$ converges so the series $\sum_{k=1}^\infty \frac{1}{k^3}$ converges.

(b) If $f(x) = \frac{1}{x\cdot\ln(x)}$, then $a_k = f(k)$ and f is continuous, positive and decreasing on $[2, \infty)$.

$$\text{Then } \int_2^\infty \frac{1}{x\cdot\ln(x)}\,dx = \lim_{n\to\infty} \int_2^n \frac{1}{x\cdot\ln(x)}\,dx = \lim_{n\to\infty} \ln(\ln(x))\Big|_2^n$$

$$= \lim_{n\to\infty} \ln(\ln(n)) - \ln(\ln(2)) = \infty.$$

The integral $\int_2^\infty \frac{1}{x\cdot\ln(x)}\,dx$ diverges so the series $\sum_{k=2}^\infty \frac{1}{k\cdot\ln(k)}$ diverges.

Practice 2: Use the Integral Test to determine whether (a) $\sum_{k=4}^\infty \frac{1}{\sqrt{k}}$ and (b) $\sum_{k=1}^\infty e^{-k}$ converge.

Note: The Integral Test does not give the value of the sum, it only answers the question of whether the series converges or diverges. Typically the value of the improper integral is not equal to the sum of the series.

P–Test for Convergence of $\sum_{k=1}^\infty \frac{1}{k^p}$

The P–Test is very easy to use. And it answers the convergence question for a whole family of series.

P–Test

The series $\sum_{k=1}^\infty \frac{1}{k^p}$ $\begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1 \end{cases}$.

Proof: If **p = 1**, then $\sum_{k=1}^{\infty} \frac{1}{k^p} = \sum_{k=1}^{\infty} \frac{1}{k}$, the harmonic series, which we already know diverges (by Section 10.3 or, using the Integral Test, since $\int_1^{\infty} \frac{1}{x}\, dx$ diverges to infinity.)

The proof for $p \neq 1$ is a straightforward application of the Integral Test on $f(x) = 1/x^p$.

If $p \neq 1$, then

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \int_1^{\infty} x^{-p}\, dx = \lim_{A\to\infty} \int_1^{A} x^{-p}\, dx$$

$$= \lim_{A\to\infty} \left(\frac{1}{1-p}\right)\cdot x^{1-p} \Big|_1^A$$

$$= \lim_{A\to\infty} \left(\frac{1}{1-p}\right)\cdot A^{1-p} - \left(\frac{1}{1-p}\right)\cdot 1$$

As we examine the limit of A^{1-p}, there are two cases to consider: $p < 1$ and $p > 1$.

If **p < 1**, then $1 - p > 0$ so A^{1-p} approaches infinity as A approaches infinity. Then $\int_1^{\infty} \frac{1}{x^p}\, dx$ diverges, so, by the Integral Test, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges.

If **p > 1**, then $p - 1 > 0$ and $A^{1-p} = \frac{1}{A^{p-1}}$ approaches 0 as A approaches infinity. Then $\int_1^{\infty} \frac{1}{x^p}\, dx$ converges, so, by the Integral Test, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.

Example 3: Use the P–Test to determine whether (a) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and (b) $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k}}$ converge.

Solution: The convergence of both series have already been determined using the Integral Test, but the P–Test is much easier to apply.

(a) $p = 2 > 1$ so $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. (b) $p = 1/2 < 1$ so $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

The P–Test is very easy to use (Is the exponent $p > 1$ or is $p \leq 1$?), and it is also very useful. In the next section we will compare new series with series whose convergence we already know, and most often this comparison is with some P–series whose convergence we know about from the P–Test.

Note: The P–Test does not give the value of the sum, it only answers the question of whether the series converges of diverges.

PROBLEMS

In problems 1 – 15 show that the function determined by the terms of the given series satisfies the hypotheses of the Integral Test, and then use the Integral Test to determine whether the series converges or diverges.

1. $\sum_{k=1}^{\infty} \frac{1}{2k+5}$

2. $\sum_{k=1}^{\infty} \frac{1}{(2k+5)^2}$

3. $\sum_{k=1}^{\infty} \frac{1}{(2k+5)^{3/2}}$

4. $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$

5. $\sum_{k=2}^{\infty} \frac{1}{k \cdot (\ln(k))^2}$

6. $\sum_{k=1}^{\infty} \frac{1}{k^2} \cdot \sin\left(\frac{1}{k}\right)$

7. $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

8. $\sum_{k=1}^{\infty} \frac{1}{k^2+100}$

9. $\sum_{k=1}^{\infty} \left\{\frac{1}{k} - \frac{1}{k+3}\right\}$

10. $\sum_{k=1}^{\infty} \left\{\frac{1}{k} - \frac{1}{k+1}\right\}$

11. $\sum_{k=1}^{\infty} \frac{1}{k(k+5)}$

12. $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$

13. $\sum_{k=1}^{\infty} k \cdot e^{-(k^2)}$

14. $\sum_{k=1}^{\infty} k^2 \cdot e^{-(k^3)}$

15. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{6k+10}}$

For problems 16 – 20, (a) use the P–Test to determine whether the given series converges, and then (b) use the Integral Test to verify your convergence conclusion of part (a).

16. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

17. $\sum_{k=1}^{\infty} \frac{1}{k^3}$

18. $\sum_{k=2}^{\infty} \frac{1}{k}$

19. $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$

20. $\sum_{k=3}^{\infty} \frac{1}{k^{2/3}}$

21. $\sum_{k=3}^{\infty} \frac{1}{k^{3/2}}$

In the proof of the Integral Test, we derived an inequality bounding the values of the partial sums $s_n = \sum_{k=1}^{n} a_k$ between the values of two integrals: $\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} a_k \le a_1 + \int_1^{n} f(x)\,dx$. For problems 22 – 27, use this inequality to determine bounds on the values of s_{10}, s_{100}, and $s_{1,000,000}$ for the given series.

22. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ (Note: The exact value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is $\frac{\pi^2}{6}$ but it beyond our means to prove that in this course.)

23. $\sum_{k=1}^{\infty} \frac{1}{k^3}$

24. $\sum_{k=1}^{\infty} \frac{1}{k}$

25. $\sum_{k=1}^{\infty} \frac{1}{k+1000}$

26. $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

27. $\sum_{k=1}^{\infty} \frac{1}{k^2+100}$

28. Euler's Constant: Define $g_1 = 1 - \ln(1) = 1$,

$g_2 = (1 + \frac{1}{2}) - \ln(2) \approx 0.806853$,

$g_3 = (1 + \frac{1}{2} + \frac{1}{3}) - \ln(3) \approx 0.734721$, and, in general,

$$g_n = (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}) - \ln(n).$$

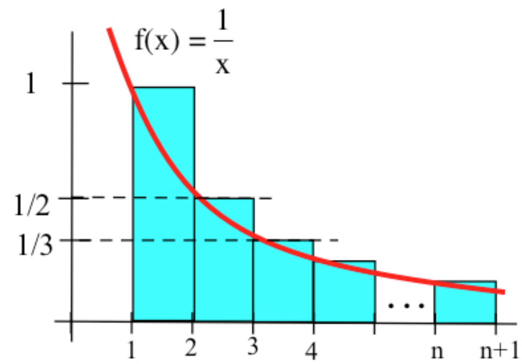

Fig. 7

(a) Make several copies of Fig. 7, and shade the regions represented by g_2, g_3, g_4, and g_n.

(b) Provide a geometric argument that $g_n > 0$ for all $n \ge 1$.

(c) Provide a geometric argument that $\{ g_n \}$ is monotonically decreasing: $g_{n+1} < g_n$ for all $n \ge 1$.

(d) Conclude from parts (b) and (c) and the Monotone Convergence Theorem (Section 10.1) that $\{ g_n \}$ converges.

(Note: The value to which $\{ g_n \}$ converges is denoted by "γ," the lower case Greek letter gamma, and is called Euler's constant. It is not even known if γ is a rational number. $\gamma \approx 0.5772157 \cdots$.)

29. (a) Show that the integral $\int_2^{\infty} \frac{1}{x \cdot (\ln x)^q}\, dx$ converges for $q>1$ and diverges for $q \le 1$.

(b) Use the result of part (a) to state a "Q test" for $\sum_{k=2}^{\infty} \frac{1}{k \cdot (\ln k)^q}$.

In problems 30 – 33, use the result of Problem 29 to determine whether the given series converge.

30. $\sum_{k=2}^{\infty} \frac{1}{k \cdot \ln(k)}$

31. $\sum_{k=2}^{\infty} \frac{1}{k \cdot (\ln k)^3}$

32. $\sum_{k=2}^{\infty} \frac{1}{k \cdot \sqrt{\ln k}}$

33. $\sum_{k=2}^{\infty} \frac{1}{k \cdot \ln(k^3)}$

Practice Answers

Practice 1: { total area of rectangular pieces } > { area under the curve in Fig. 3a } so if the shaded area in Fig. 3a is infinite, then the shaded area in Fig. 3b is also infinite.

Practice 2: (a) Let $f(x) = \frac{1}{\sqrt{x}}$. Then $a_k = f(k)$ and f is continuous, positive and decreasing on $[1, \infty)$.

Then $$\int_4^\infty \frac{4}{\sqrt{x}}\, dx = \lim_{n\to\infty} \int_1^n \frac{1}{\sqrt{x}}\, dx = \lim_{n\to\infty} 2\, x^{1/2} \Big|_4^n$$
$$= \lim_{n\to\infty} 2\sqrt{n} - 2\sqrt{4} = \infty .$$

The integral $\int_4^\infty \frac{1}{\sqrt{x}}\, dx$ diverges so the series $\sum_{k=4}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

(Note: It will be easier to determine that this series diverges by using the P–Test which occurs right after this Practice problem in the text.)

(b) Let $f(x) = e^{-x}$. Then $e^{-k} = a_k = f(k)$ and f is continuous, positive and decreasing on $[1, \infty)$.

Then $$\int_1^\infty e^{-x}\, dx = \lim_{n\to\infty} \int_1^n e^{-x}\, dx = \lim_{n\to\infty} -e^{-x}\Big|_1^n = \lim_{n\to\infty} \left(-e^{-n}\right) - \left(-e^{-1}\right)$$
$$= \lim_{n\to\infty} \left(-\frac{1}{e^n}\right) - \left(-\frac{1}{e}\right) = \frac{1}{e} \approx 0.368 .$$

The integral $\int_1^\infty e^{-x}\, dx$ converges so the series $\sum_{k=1}^{\infty} e^{-k}$ converges.