10.1 SEQUENCES

Sequences play important roles in several areas of theoretical and applied mathematics. As you study additional mathematics you will encounter them again. In this course, however, their role is primarily as a foundation for our study of series ("big polynomials"). In order to understand how and where it is valid to represent a function such as sine as a series, we need to examine what it means to add together an infinite number of values. And in order to understand this infinite addition we need to analyze lists of numbers (called sequences) and determine whether or not the numbers in the list are converging to a single value. This section examines sequences, how to represent sequences graphically, what it means for a sequence to converge, and several techniques to determine if a sequence converges.

Example 1: A person places \$100 in an account that pays 8% interest at the end of each year. How much will be in the account at the end of 1 year, 2 years, 3 years, and n years?

Solution: After one year, the total is the principal plus the interest: $100 + (.08)100 = \mathbf{(1.08)\cdot 100} = \$108$.

At the end of the second year, the amount is 108% of the amount at the start of the second year:

$$(1.08)\,\{\,(1.08)100\,\} = \mathbf{(1.08)^2 \cdot 100} = \$116.64\ .$$

At the end of the third year, the amount is 108% of the amount at the start of the third year:

$$(1.08)\,\{\,(1.08)^2\,100\,\} = \mathbf{(1.08)^3 \cdot 100} = \$125.97\ .$$

These results are shown in Fig. 1. In general, at the end of the n^{th} year, the amount in the account is $\mathbf{(1.08)^n \cdot 100}$ dollars.

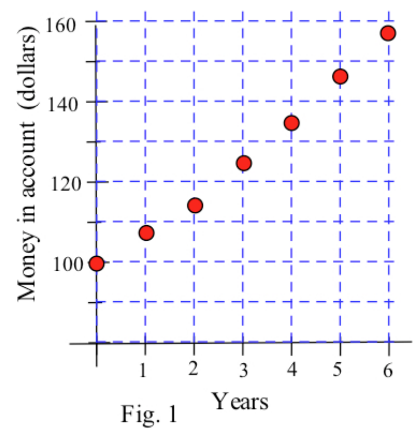


Fig. 1

Practice 1: A layer of protective film transmits two thirds of the light that reaches that layer. How much of the incoming light is transmitted through 1 layer, 2 layers, 3 layers, and n layers? (Fig. 2)

The Example and Practice each asked for a list of numbers in a definite order: a first number, then a second number, and so on. Such a list of numbers in a definite order is called a **sequence**. An infinite sequence is one that just keeps going and has no last number. Often the pattern of a sequence is clear from the first few numbers, but in order to precisely specify a sequence, a rule for finding the value of the n^{th} term, a_n ("a sub n"), in the sequence is usually given.

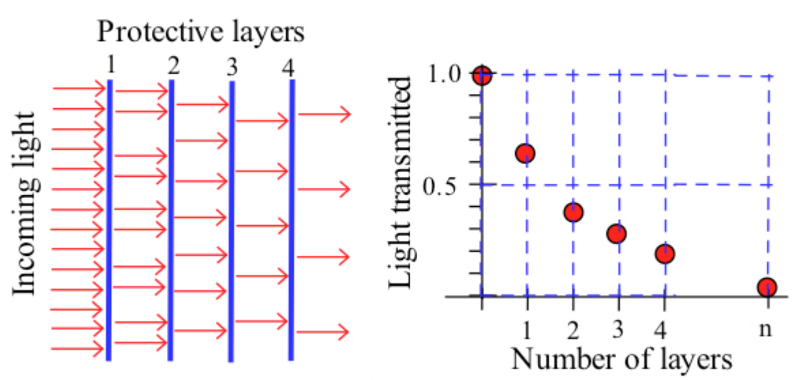


Fig. 2

Example 2: List the next two numbers in each sequence and give a rule for calculating the n^{th} number, a_n:

(a) $1, 4, 9, 16, \ldots$ (b) $-1, 1, -1, 1, \ldots$ (c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$.

Solution: (a) $a_5 = 25, a_6 = 36$, and $a_n = n^2$. (b) $a_5 = -1, a_6 = 1$, and $a_n = (-1)^n$.

(c) $a_5 = \frac{1}{32}$, $a_6 = \frac{1}{64}$, and $a_n = (\frac{1}{2})^n = \frac{1}{2^n}$.

Practice 2: List the next two numbers in each sequence and give a rule for calculating the n^{th} number, a_n:

(a) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (b) $\frac{-1}{2}, \frac{1}{4}, \frac{-1}{8}, \frac{1}{16}, \ldots$ (c) $2, 2, 2, 2, \ldots$

Definition and Notation

Since a sequence gives a single value for each integer n, a sequence is a function, but a function whose domain is restricted to the integers.

Definition

A **sequence** is a function whose domain is all integers greater than or equal to a starting integer.

Most of our sequences will have a starting integer of 1, but sometimes it is convenient to start with 0 or another integer value.

Notation: The symbol a_n represents a single number called the n^{th} term .

The symbol $\{ a_n \}$ represents the entire sequence of numbers, the set of all terms.

The symbol $\{ \textit{rule} \}$ represents the sequence generated by the rule.

The symbol $\{ a_n \}_{n=3}$ represents the sequence that starts with n = 3.

Because sequences are functions, we can add, subtract, multiply, and divide them, and we can combine them with other functions to form new sequences. We can also graph sequences, and their graphs can sometimes help us describe and understand their behavior.

Example 3: For the sequences given by $a_n = 3 - \frac{1}{n}$ and $b_n = \frac{1}{2^n}$, graph the points (n, a_n) and (n, b_n) for n = 1 to 5. Calculate the first 5 terms of $c_n = a_n + b_n$ and graph the points (n, c_n).

Solution: $c_1 = (3 - \frac{1}{1}) + (\frac{1}{2^1}) = 2.5, c_2 = 2.75, c_3 \approx 2.792$, $c_4 = 2.8125, c_5 = 2.83125$. The graphs of (n, a_n), (n, b_n), and (n, c_n) are shown in Fig. 3.

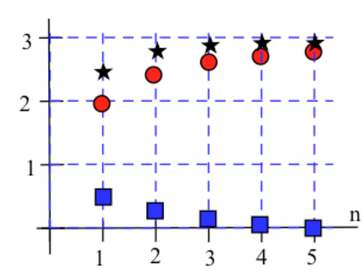


Fig. 3

Practice 3: For a_n and b_n in the previous example, calculate the first 5 terms of $c_n = a_n - b_n$ and $d_n = (-1)^n b_n$ and graph the points (n, c_n) and (n, d_n) .

Recursive Sequences

A recursive sequence is a sequence defined by a rule that gives each new term in the sequence as a combination of some of the previous terms. We already encountered a recursive sequence when we studied Newton's Method for approximating roots of a function (Section 2.7). Newton's method for finding the roots of a function generates a recursive sequence $\{ x_1, x_2, x_3, x_4, ...\}$, as do successive iterations of a function and other operations.

Example 4: Let $f(x) = x^2 - 4$. Take $x_1 = 3$ and apply Newton's method (Section 2.7) to calculate x_2 and x_3. Give a rule for x_n.

Solution: $f(x) = x^2 - 4$ so $f'(x) = 2x$, and, by Newton's method,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{5}{6} = \frac{13}{6} \approx 2.1667.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{13}{6} - \frac{f(13/6)}{f'(13/6)} = \frac{13}{6} - \frac{25}{156} = \frac{313}{156} \approx 2.0064 .$$

In general, $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})} = x_{n-1} - \dfrac{(x_{n-1})^2 - 4}{2x_{n-1}}$.

The terms $x_1, x_2, ...$ approach the value 2, one solution of $x^2 - 4 = 0$. The sequence $\{ x_n \}$ is a recursive sequence since each term x_n is defined as a function of the previous term x_{n-1}.

Practice 4: Let $f(x) = 2x - 1$, and define $a_n = f(f(f(\ldots f(a_0) \ldots)))$ where the function is applied n times. Put $a_0 = 3$ and calculate a_1, a_2, and a_3. Note that a_n can be defined recursively as $a_n = f(a_{n-1})$.

Example 5: Let $a_n = 1/2^n$, and define a second sequence $\{ s_n \}$ by the rule that s_n is the **sum** of the first n terms of a_n. Calculate the values of s_n for n = 1 to 5.

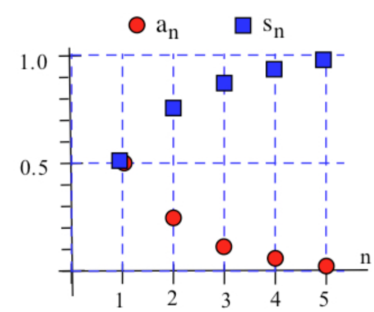


Fig. 4

Solution: $s_1 = a_1 = 1/2$, $s_2 = a_1 + a_2 = 1/2 + 1/4 = 3/4$,
$s_3 = a_1 + a_2 + a_3 = 1/2 + 1/4 + 1/8 = 7/8$, $s_4 = 15/16$,
and $s_5 = 31/32$. (Fig. 4)

You should notice two patterns in these sums.
First, it appears that $s_n = (2^n - 1)/2^n$.
Second, you can simplify the addition process: each term s_n is the sum of the previous term s_{n-1} and the a_n term: $s_n = s_{n-1} + a_n$. We will meet this second pattern again in the next section.

Practice 5: Let $b_0 = 0$ and, for $n > 0$, define $b_n = b_{n-1} + 1/3^n$. Calculate b_n for n = 1 to 4.

Limits of Sequences: Convergence

Since sequences are discrete functions defined only on integers, some calculus ideas for continuous functions are not applicable to sequences. One type of limit, however, is used: the limit as n approaches infinity. Do the values a_n eventually approach (or equal) some number?

We say that the limit of a sequence $\{ a_n \}$ is L if the terms a_n are arbitrarily close to L for sufficiently large values of n — the terms at the beginning of the sequence can be any values, but for large values of n, the a_n terms are all close to L. The following definition puts this idea more precisely.

Definition

$\lim\limits_{n\to\infty} \mathbf{a_n} = \mathbf{L}$ if for any $\varepsilon > 0$ ("epsilon > 0")

there is an index N (typically depending on ε)

so that a_n is within ε of L whenever

n is larger than N :

$n > N$ implies $| a_n - L | < \varepsilon$ (Fig. 5)

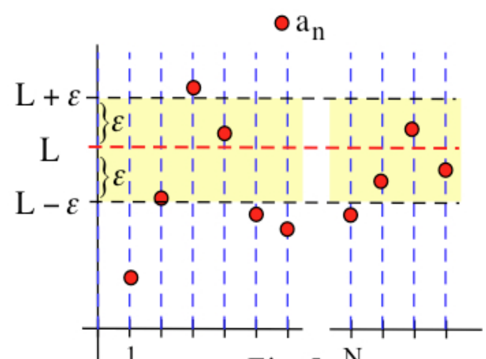


Fig. 5

If a sequence has a finite limit L, we say that the sequence "converges to L." If a sequence does not have a finite limit, we say the sequence "diverges." Typically a sequence diverges because its terms grow infinitely large (positively or negatively) or because the terms oscillate and do not approach a single number.

Example 7: For $a_n = 3 + 1/n^2$ show that $\lim\limits_{n\to\infty} a_n = 3$.

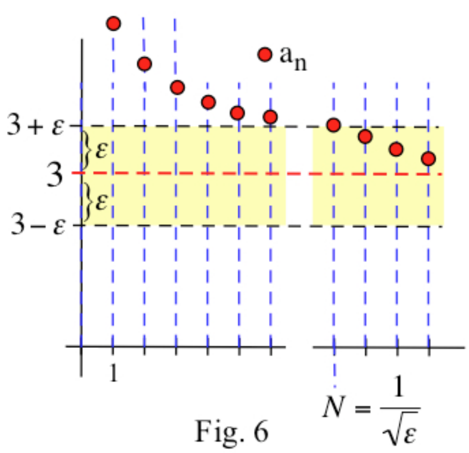


Fig. 6

Solution: We need to show that for any positive ε, there is a number N so that the distance from a_n to L, $| a_n - L |$, is less than ε whenever n is larger than N. For this particular sequence and limit we need to show that for any positive ε, there is a number N so that (Fig. 6)

$$| (3 + 1/n^2) - 3 | < \varepsilon \text{ whenever } n > N.$$

To determine what N might be, we solve the inequality $| (3 + 1/n^2) - 3 | < \varepsilon$ for n in terms of ε:

$$| 1/n^2 | < \varepsilon \quad \text{so} \quad 1/\varepsilon < n^2 \quad \text{and} \quad n > 1/\sqrt{\varepsilon} \;.$$

So for **any** positive ε, we can take $N = 1/\sqrt{\varepsilon}$ (or the next larger integer). Then for $n > N$ we know that

$$n > 1/\sqrt{\varepsilon} \quad \text{so} \quad 1/n^2 < \varepsilon \text{ and } | (3 + 1/n^2) - 3 | < \varepsilon \;.$$

Practice 6: For $a_n = (n + 1)/n$ show that $\lim\limits_{n\to\infty} a_n = 1$. (Fig. 7)

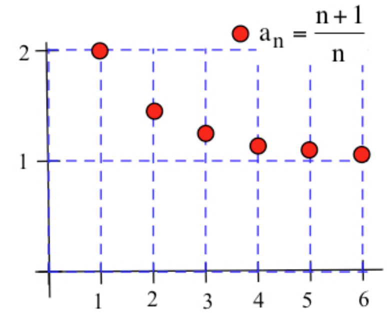


Fig. 7

The limit of a sequence, as n approaches infinity, depends only on the behavior of the terms of the sequence for large values of n (the "tail end") and not on the values of the first few (or few thousand) terms. As a consequence, we can insert or delete any **finite** number of terms without changing the convergence behavior of the sequence.

Sequences are functions, so limits of sequences share many properties with limits of other functions, and we state only a few of them.

Uniqueness Theorem

If a sequence converges to a limit, then the limit is unique.

A sequence can not converge to two different values.

A proof of the Uniqueness Theorem is given after the problem set.

Sometimes it is useful to replace a sequence $\{ a_n \}$, a function whose domain is integers, with a function f whose domain is the real numbers so $a_n = f(n)$. If f(x) has a limit as "$x \to \infty$," as x gets arbitrarily large, then f(n) has the same limit as "$n \to \infty$" (Fig. 8). This replacement of "x" with "n" allows us to use earlier results about functions, particularly L'Hopital's Rule, to calculate limits of sequences.

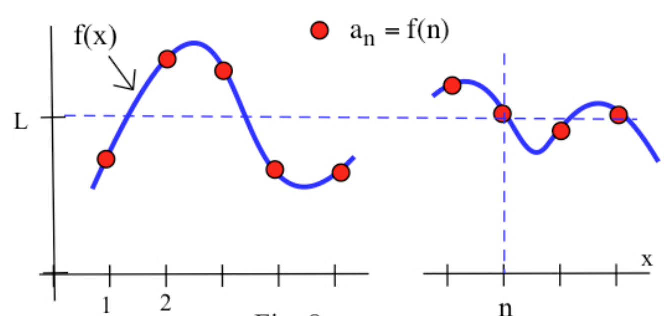


Fig. 8

Theorem: If $a_n = f(n)$ and $\lim\limits_{x\to\infty} f(x) = L$

then $\{ a_n \}$ converges to L: $\lim\limits_{n\to\infty} a_n = L$

Example 8: Calculate $\lim\limits_{n\to\infty} \left(1+\frac{2}{n}\right)^n$.

Solution: The terms of the sequence are $a_n = \left(1 + \frac{2}{n} \right)^n$, so we can define $f(x) = \left(1 + \frac{2}{x} \right)^x$ by replacing the integer values n with real number values x. Then $a_n = f(n)$, and we can use

L'Hopital's rule to get $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = e^2$ (Section 3.7, Example 6). Finally, we can conclude that

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = e^2 \approx 7.389 \quad \text{(Fig. 9)}.$$

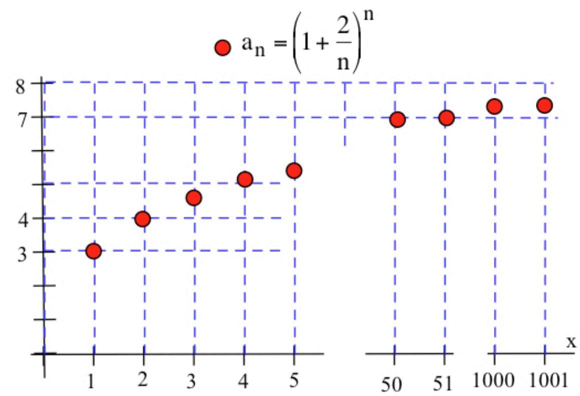


Fig. 9

Practice 7: Calculate $\lim_{n\to\infty} \frac{\ln(n)}{n}$.

A **subsequence** is an infinite set of terms from a sequence that occur in the same order as they appear in the original sequence. The sequence of even integers { 2, 4, 6, . . . } is a subsequence of the sequence of all positive integers { 1, **2**, 3, **4**, . . . }. The sequence of reciprocals of primes { 1/2, 1/3, 1/5, 1/7, . . . } is a subsequence of the sequence of the reciprocals of all positive integers { 1, **1/2**, **1/3**, 1/4, **1/5**, . . . }. Subsequences inherit some properties from their original sequences.

Subsequence Theorem

Every subsequence of a convergent sequence converges to the same limit as the original sequence:

if $\lim_{n\to\infty} a_n = L$ and $\{ b_n \}$ is a subsequence of $\{ a_n \}$,

then $\lim_{n\to\infty} b_n = L$. (Fig. 10)

If the sequence $\{ a_n \}$ does not converge, then the subsequence $\{ b_n \}$ may or may not converge.

Corollary: If two subsequences of the same sequence converge to two different limits, then the original sequence diverges.

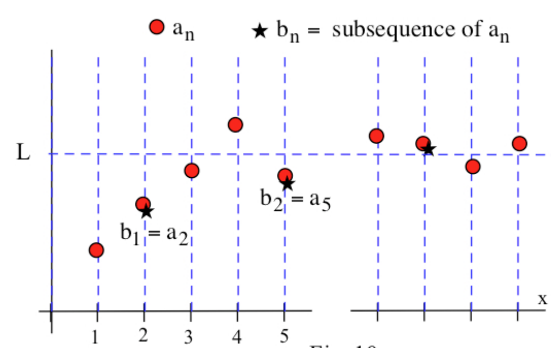


Fig. 10

Example 9: Show that the sequence $\left\{ \frac{(-1)^n n}{n+1} \right\}$ diverges.

Solution: If n is even, then the even terms $a_n = \frac{(-1)^{\text{even}} n}{n+1} = \frac{n}{n+1}$ converge to +1 so the subsequence of even terms converges to 1.

If n is odd, then the odd terms $a_n = \frac{(-1)^{\text{odd}} n}{n+1} = \frac{-n}{n+1}$ converge to –1 so the subsequence of odd terms converges to –1. Finally, since the two subsequences converge to different values, we can conclude that the original sequence $\left\{ \frac{(-1)^n n}{n+1} \right\}$ diverges.

Practice 8: Show that the sequence $\{ \sin(n\pi/2) \}$ diverges.

Bounded and Monotonic Sequences

A sequence $\{ a_n \}$ is **bounded above** if there is a value A so that $a_n \le A$ for all values of n: A is called an **upper bound** of the sequence (Fig. 11). Similarly, $\{ a_n \}$ is **bounded below** if there is a value B so that $B \le a_n$ for all n: B is called a **lower bound** of the sequence. A sequence is **bounded** if it has an upper bound and a lower bound. All of the terms of a bounded sequence are between (or equal to) the upper and lower bounds (Fig. 12)

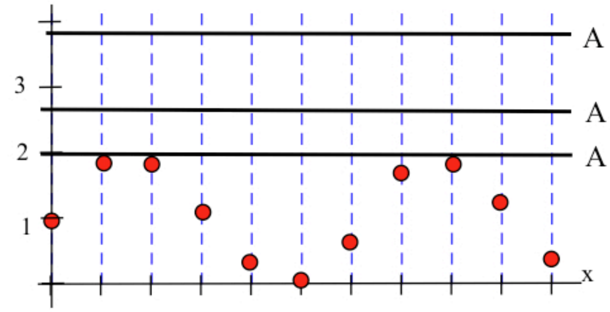


Fig. 11: Several upper bounds of the sequence $a_n = 1 + \sin(n)$

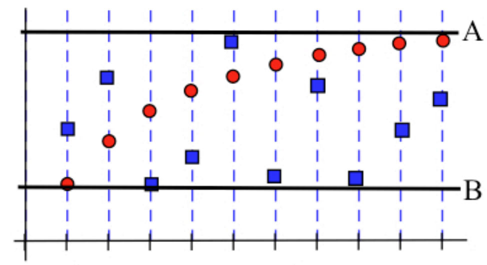
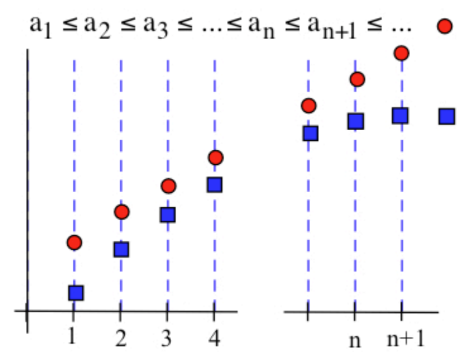


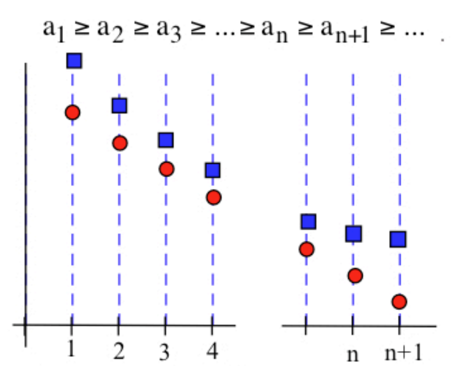
Fig. 12: Two bounded sequences

A **monotonically increasing** sequence is a sequence in which each term is greater than or equal to the previous term, $a_1 \le a_2 \le a_3 \le \ldots$ (Fig. 13); a **monotonically decreasing** sequence is one in which each term is less than or equal to the previous term, $a_1 \ge a_2 \ge a_3 \ge \ldots$ (Fig. 14). A monotonic sequence does not oscillate: if one term is larger than a previous term and another term is smaller than a previous term, then the sequence is not monotonic increasing or decreasing.



Two monotonic increasing sequences

Fig. 13



Two monotonic decreasing sequences

Fig. 14

There are three basic ways to show that a sequence is monotonically increasing:

(i) by showing that $a_{n+1} \ge a_n$ for all n,

(ii) by showing that all the a_n are positive and

$$\frac{a_{n+1}}{a_n} \ge 1 \text{ for all n, or}$$

(iii) by showing that $a_n = f(n)$ for integer values n

and $f'(x) \ge 0$ for all $x > 0$.

Practice 9: List three ways you can show that a sequence is monotonically decreasing.

Example 10: Show that the sequence $a_n = \dfrac{2^n}{n!}$ is monotonically decreasing by showing that $\dfrac{a_{n+1}}{a_n} \le 1$ for all n.

Solution: $a_n = \frac{2^n}{n!} = \frac{2^n}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}$ and $a_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2^n \cdot 2}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1)}$.

Then $\frac{a_{n+1}}{a_n} = \frac{2^n \cdot 2}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1)} \cdot \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{2^n}$ by inverting and multiplying

$= \frac{2^n \cdot 2}{2^n} \cdot \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1)}$ by reorganizing the top and bottom

$= \frac{2}{n+1} \le 1$ for all positive integers n.

Practice 10: Show that $\{ (\frac{2}{3})^n \}$ is monotonically decreasing.

Since the behavior of a monotonic sequence is so regular, it is usually easy to determine if a monotonic sequence has a finite limit: all we need to do is show that it is bounded.

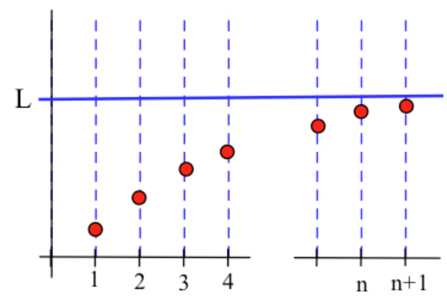


Fig. 15

Monotone Convergence Theorem

If a monotonic sequence is bounded,

then the sequence converges. (Fig. 15)

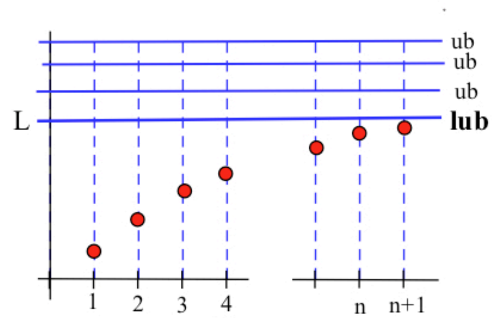


Fig. 16

Idea for a proof for a monotonically increasing sequence that is bounded above:

If the sequence $\{ a_n \}$ is bounded above, then $\{ a_n \}$ has an infinite number of upper bounds (Fig. 16), and each of these upper bounds is larger than every a_n. If L is the smallest of the upper bounds (the least upper bound) of $\{ a_n \}$, then there is a value a_N as close as we want to L (otherwise there would be an upper bound smaller than L). Finally, if a_N is close to L, then the later values, a_n with $n \ge N$, are even closer to L because $\{ a_n \}$ is monotonically increasing, so L is the limit of $\{ a_n \}$.

Cauchy and other mathematicians accepted this theorem on intuitive and geometric grounds similar to the "idea for a proof" given above, but later mathematicians felt more rigor was needed. However, even the mathematician Dedekind who supplied much of that rigor recognized the usefulness of geometric intuition.

> "*Even now such resort to geometric intuition in a first presentation of differential calculus, I regard as exceedingly useful, from a didactic standpoint, and indeed indispensable if one does not wish to lose too much time.*" (Dedekind, Essays on the Theory of Numbers, 1901 (Dover, 1963), pp. 1–2)

PROBLEMS In problems 1 – 6, find a rule which describes the given numbers in the sequence.

1. 1, 1/4, 1/9, 1/16, 1/25, . . .
2. 1, 1/8, 1/27, 1/64, 1/125, . . .
3. 0, 1/2, 2/3, 3/4, 4/5, . .
4. –1, 1/3, –1/9, 1/27, –1/81, . . .
5. 1/2, 2/4, 3/8, 4/16, 5/32, . . .
6. 7, 7, 7, 7, 7, . . .

(Bonus: O, T, T, F, F, S, S, E, ? , ?)

In problems 7 – 18, calculate the first 6 terms (starting with n = 1) of each sequence and graph these terms.

7. $\{ 1 - \frac{2}{n} \}$
8. $\{ 3 + \frac{1}{n^2} \}$
9. $\{ \frac{n}{2n-1} \}$
10. $\{ \frac{\ln(n)}{n} \}$
11. $\{ 3 + \frac{(-1)^n}{n} \}$
12. $\{ 4 + (-1)^n \}$
13. $\{ (-1)^n \frac{n-1}{n} \}$
14. $\{ \cos(n\pi/2) \}$
15. $\{ \frac{1}{n!} \}$
16. $\{ \frac{n+1}{n!} \}$
17. $\{ \frac{2^n}{n!} \}$
18. $\{ (1 + \frac{1}{n})^n \}$

In problems 19 – 24, calculate the first 10 terms (starting with n = 1) of each sequence.

19. $a_1 = 2$ and $a_{n+1} = -a_n$
20. $b_1 = 3$ and $b_{n+1} = 1/b_n$
21. $\{ \sin(\frac{2\pi n}{3}) \}$
22. $a_1 = 2, a_2 = 3$, and, for $n \ge 3$, $a_n = a_{n-1} - a_{n-2}$
23. c_n = the sum of the first n positive integers
24. d_n = the sum of the first n prime numbers (2 is the first prime)

In problems 25 – 28, state whether each sequence appears to be converging or diverging. If you think the sequence is converging, mark its limit as a value on the vertical axis. (***Important Note:*** *The behavior of a sequence can change drastically after awhile, and the first terms have* ***no*** *influence on whether or not the sequence converges. However, sometimes the first few terms are the only values we have, and we need to reach a* ***tentative*** *conclusion based on those values*.)

25. Sequences A and B in Fig. 17.
26. Sequences C and D in Fig. 18.

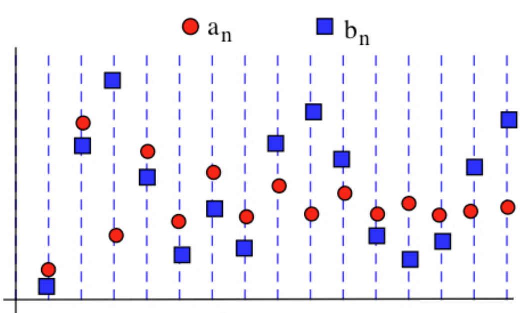


Fig. 17

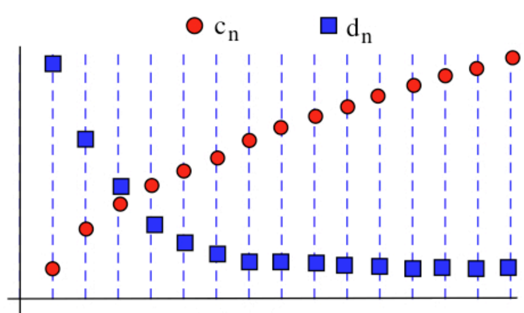


Fig. 18

27. Sequences E and F in Fig. 19.

28. Sequences G and H in Fig. 20.

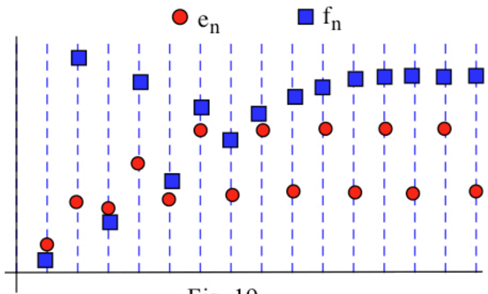


Fig. 19

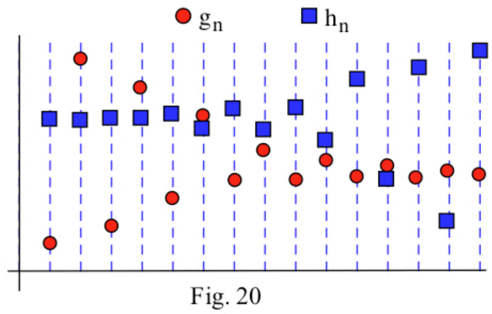


Fig. 20

In problems 29 – 43, state whether each sequence converges or diverges. If the sequence converges, find its limit.

29. $\{ 1 - \frac{2}{n} \}$ 30. $\{ n^2 \}$ 31. $\{ \frac{n^2}{n+1} \}$ 32. $\{ 3 + \frac{1}{n^2} \}$

33. $\{ \frac{n}{2n-1} \}$ 34. $\{ \frac{\ln(n)}{n} \}$ 35. $\{ \ln(3 + \frac{7}{n}) \}$ 36. $\{ 3 + \frac{(-1)^{n+1}}{n} \}$

37. $\{ 4 + (-1)^n \}$ 38. $\{ (-1)^n \frac{n-1}{n} \}$ 39. $\{ \frac{1}{n!} \}$ 40. $\{ (1 + \frac{3}{n})^n \}$

41. $\{ (1 - \frac{1}{n})^n \}$ 42. $\{ \frac{\sqrt{n}-3}{\sqrt{n}+3} \}$ 43. $\{ \frac{(n+2)(n-5)}{n^2} \}$

In problems 44 – 47, prove that the sequence converges to the given limit by showing that for any $\varepsilon > 0$, you can find an N which satisfies the conditions of the definition of convergence.

44. $\lim\limits_{n\to\infty} 2 - \frac{3}{n} = 2$ 45. $\lim\limits_{n\to\infty} \frac{3}{n^2} = 0$

46. $\lim\limits_{n\to\infty} \frac{7}{n+1} = 0$ 47. $\lim\limits_{n\to\infty} \frac{3n-1}{n} = 3$

In problems 48 – 53, use subsequences to help determine whether the sequence converges or diverges. If the sequence converges, find its limit.

48. $\{ (-1)^n 3 \}$ 49. $\{ \frac{1}{n^{\text{th}} \text{ prime}} \}$ 50. $\{ (-1)^n \frac{n+1}{n} \}$

51. $\{ (-2)^n (\frac{1}{3})^n \}$ 52. $\{ (1 + \frac{1}{3n})^{3n} \}$ 53. $\{ (1 + \frac{5}{n^2})^{(n^2)} \}$

In problems 54 – 58, calculate $a_{n+1} - a_n$ and use that value to determine whether $\{ a_n \}$ is monotonic increasing, monotonic decreasing, or neither.

54. $\{ \frac{3}{n} \}$ 55. $\{ 7 - \frac{2}{n} \}$ 56. $\{ \frac{n-1}{2n} \}$ 57. $\{ 2^n \}$ 58. $\{ 1 - \frac{1}{2^n} \}$

In problems 59 – 63, calculate a_{n+1} / a_n and use that value to determine whether each sequence is monotonic increasing, monotonic decreasing, or neither.

59. $\{ \frac{n+1}{n!} \}$ 60. $\{ \frac{n}{n+1} \}$ 61. $\{ (\frac{5}{4})^n \}$ 62. $\{ \frac{n^2}{n!} \}$ 63. $\{ \frac{n}{e^n} \}$

In problems 64 – 68, use derivatives to determine whether each sequence is monotonic increasing, monotonic decreasing, or neither.

64. $\{ \frac{n+1}{n} \}$ 65. $\{ 5 - \frac{3}{n} \}$ 66. $\{ n \cdot e^{-n} \}$ 67. $\{ \cos(1/n) \}$ 68. $\{ (1 + \frac{1}{n})^3 \}$

In problems 69 – 73, show that each sequence is monotonic.

69. $\{ \frac{n+3}{n!} \}$ 70. $\{ \frac{n}{n+1} \}$ 71. $\{ 1 - \frac{1}{2^n} \}$ 72. $\{ \sin(1/n) \}$ 73. $\{ \frac{n+1}{e^n} \}$

74. The Fibonacci sequence (after Leonardo Fibonacci (1170–1250) who used it to model a population of rabbits) is obtained by setting the first two terms equal to 1 and then defining each new term as the sum of the two previous terms: $a_n = a_{n-1} + a_{n-2}$ for $n \ge 3$. (a) Write the first 7 terms of this sequence. (b) Calculate the successive ratios of the terms, a_n/a_{n-1} . (These ratios approach the "golden mean," approximately 1.618)

75. Heron's method for approximating roots: To approximate the square root of a positive number N, put $a_1 = N$ and let $a_{n+1} = \frac{1}{2}(a_n + \frac{N}{a_n})$. Then $\{ a_n \}$ converges to $\sqrt{N}$. Calculate a_1 through a_4 for N = 4, 9, and 5. (Heron's method is equivalent to Newton's method applied to the function $f(x) = x^2 - N$.)

76. Hailstone Sequence For the initial or "seed" value h_0, define the hailstone sequence by the rule

$$h_n = \begin{cases} 3 \cdot h_{n-1} + 1 & \text{if } h_{n-1} \text{ is odd} \\ \frac{1}{2} \cdot h_{n-1} & \text{if } h_{n-1} \text{ is even} \end{cases} .$$

Define the **length** of the sequence to be the first value of n so that $h_n = 1$. If the seed value is $h_0 = 3$, then $h_1 = 3(3) + 1 = 10$, $h_2 = (10)/2 = 5$, $h_3 = 3(5) + 1 = 16$, $h_4 = 16/2 = 8$, $h_5 = 4$, $h_6 = 2$, and $h_7 = 1$ so the **length** of the hailstone sequence is 7 for the seed value $h_0 = 3$.

(a) Find the length of the hailstone sequence for each seed value from 2 to 10.

(b) Find the length of the hailstone sequence for a seed value $h_0 = 2^n$.

** (c) Open question (no one has been able to answer the this question): Is the length of the hailstone sequence finite for every seed value?

(This is called the hailstone sequence because for some seed values, the terms of the sequence rise and drop just like the path of a hailstone as it forms. This sequence is attributed to Lothar Collatz and (c) is also called Ulam's conjecture, Syracuse's problem, Kakutani's problem and Hasse's algorithm. "The 3n+1 sequence has probably consumed more CPU time than any other number theoretic conjecture," says Gaston Gonnett of Zurich.)

77. Negative Eugenics: Suppose that individuals with the gene combination "aa" do not reproduce and those with the combinations "aA" and "AA" do reproduce. When the initial proportion of individuals with "aa" is $a_0 = p$ (typically a small number), then the proportion of individuals with "aa" in the k^{th} generation is $a_k = \frac{p}{kp + 1}$. Use this formula for a_k to answer the following questions.

(a) If 2% of a population initially have the combination "aa" and these individuals do not reproduce, then how many generations will it take for the proportion of individuals with "aa" to drop to 1% ?

(b) In general, find the number of generations until the proportion of individuals with "aa" is half of the initial proportion.

("Negative eugenics" is a strategy in which individuals with an undesirable trait are prevented from reproducing. It is not an effective strategy for traits carried by recessive genes (the above example) which are uncommon (p small) in a species which reproduces slowly (people). Mathematics shows that the social strategy of sterilizing people with some undesirable trait, as proposed in the early 20th century, won't effectively reduce the trait in the population.)

78. The fractional part of a number is the number minus its integer part: $x - \text{INT}(x)$. The sequence of fractional parts of multiples of a number x is the sequence with terms $a_n = n \cdot x - \text{INT}(n \cdot x)$. The behavior of the sequence of fractional parts of the multiples of a number is one way in which rational numbers differ from irrational numbers.

(a) Let $a_n = nx - \text{INT}(nx)$ be the fractional part of the n^{th} multiple of x . Calculate a_1 through a_6 for $x = 1/3$. These are the fractional parts of the first 6 multiples of 1/3.

(b) Calculate the fractional parts of the first 9 multiples of 3/4, and 2/5 .

(c) Calculate the fractional parts of the first 5 multiples of π.

* (d) Let $a_n = n \cdot \pi - \text{INT}(n \cdot \pi)$ be the fractional part of the n^{th} multiple of π. Is it possible for two different multiples of π to have the same fractional part? (Suggestion: Assume the answer is yes and derive a contradiction. Assume that $a_n = a_m$ for some $m \neq n$, and derive the contradiction that $\pi = \frac{\text{INT}(n\pi) - \text{INT}(m\pi)}{n-m}$. Why is this a contradiction?)

An Alternate Way to Visualize Sequences and Convergence

A sequence is a function, and we have graphed sequences in the xy plane in the same way we graphed other functions: since $a_n = f(n)$, we plotted the point (n, a_n). If the sequence $\{ a_n \}$ converges to L, then the points (n, a_n) are eventually (for big values of n) close to or on the horizontal line $y = L$.

We can also graph a sequence $\{ a_n \}$ in one dimension, on the x–axis. For each value of n, we plot the point $x = a_n$. Then the graph of $\{ a_n \}$ consists of a collection of points on the x–axis. Fig. 21 shows the one dimensional graphs of $a_n = \frac{1}{n}$, $b_n = 2 + \frac{(-1)^n}{n}$, and $c_n = (-1)^n$.

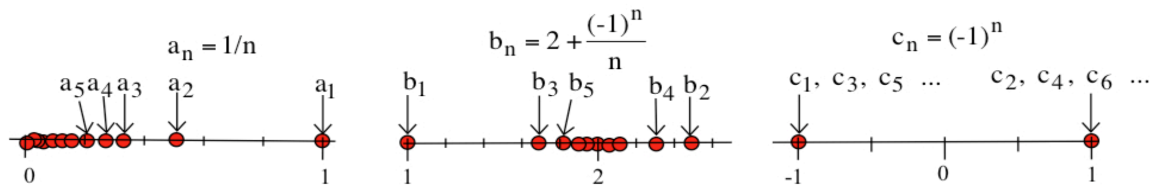


Fig. 21

If $\{ a_n \}$ converges to L, then the points $x = a_n$ are eventually (for big values of n) close to or on the point $x = L$. If we build a narrow box, with width $2\varepsilon > 0$, and center the box at the point $x = L$, then all of the points a_n will fall into the box once n is larger than some value N.

79. Suppose that the sequence $\{ a_n \}$ converges to 3 and that you place a single grain of sand at each point $x = a_n$ on the x–axis. Describe the likely result (a) after a few grains have been placed and (b) after a lot (thousands or millions) of grains have been placed.

80. Suppose the sequence $\{ a_n \}$ converges to 3, $\{ b_n \}$ converges to 1, and that you place a single grain of sand at each point $(x, y) = (a_n, b_n)$ on the xy–plane. Describe the likely result (a) after a few grains have been placed and (b) after a lot (thousands or millions) of grains have been placed.

81. Suppose that $a_n = \sin(n)$ for positive integers n. If you place a single grain of sand at each point $x = a_n$ on the x–axis. Describe the likely result (a) after a few grains have been placed and (b) after a lot (thousands or millions) of grains have been placed. (c) Do two grains ever end up on the same point?

82. Suppose that $a_n = \cos(n)$, and $b_n = \sin(n)$ for positive integers n. If you place a single grain of sand at each point $(x, y) = (a_n, b_n)$ on the xy–plane. Describe the likely result (a) after a few grains have been placed and (b) after a lot (thousands or millions) of grains have been placed.

Practice Answers

Practice 1: One layer transmits 2/3 of the original light. Two layers transmit $(\frac{2}{3})(\frac{2}{3}) = (\frac{2}{3})^2$ of the original light. Three layers transmit $(\frac{2}{3})^3$, and, in general, n layers transmit $(\frac{2}{3})^n$ of the original light.

Practice 2: (a) $1, 1/2, 1/3, 1/4, 1/5, 1/6, \ldots, 1/n, \ldots$

(b) $-1/2, 1/4, -1/8, 1/16, -1/32, 1/64, \ldots, (-1/2)^n$ or $(-1)^n (1/2)^n, \ldots$

(c) $2, 2, 2, 2, 2, \ldots, 2, \ldots$

Practice 3: $c_n = a_n - b_n$: $c1 = (3 - \frac{1}{1}) - (\frac{1}{2}) = \frac{3}{2} = 1.5$, $c_2 = (3 - \frac{1}{2}) - (\frac{1}{2^2}) = \frac{9}{4} = 2.25$,

$c_3 = (3 - \frac{1}{3}) - (\frac{1}{2^3}) = \frac{61}{24} \approx 2.542$, $c_4 = (3 - \frac{1}{4}) - (\frac{1}{2^4}) = \frac{43}{16} \approx 2.687$,

$c_5 = (3 - \frac{1}{5}) - (\frac{1}{2^5}) = \frac{443}{160} \approx 2.769$

$d_n = (-1)^n b_n$: $d_1 = (-1)^1(\frac{1}{2}) = -\frac{1}{2}$, $d_2 = (-1)^2(\frac{1}{2^2}) = \frac{1}{4}$, $d_3 = (-1)^3(\frac{1}{2^3}) = -\frac{1}{8}$,

$d_4 = \frac{1}{16}$, $d_2 - \frac{1}{32}$

Practice 4: $a_0 = 3$: $a_1 = f(a_0) = 2(3) - 1 = 5$, $a_2 = f(a_1) = 2(5) - 1 = 9$, $a_3 = f(a_2) = 2(9) - 1 = 17$

Practice 5: $b_1 = b_0 + \frac{1}{3} = 0 + \frac{1}{3} = \frac{1}{3}$, $b_2 = b_1 + \frac{1}{9} = \frac{4}{9}$, $b_3 = b_2 + \frac{1}{27} = \frac{13}{27}$, $b_4 = b_3 + \frac{1}{81} = \frac{40}{81}$

Practice 6: For $a_n = (n + 1)/n$ show that $\lim_{n\to\infty} a_n = 1$:

We need to show that for any positive ε, there is a number N so that the distance from $a_n = \frac{n+1}{n}$ to L, $|a_n - L|$, is less than ε whenever n is larger than N.

For this particular $|a_n - L| = |\frac{n+1}{n} - 1| = |\frac{n}{n} + \frac{1}{n} - 1| = |\frac{1}{n}|$.

To determine what N might be, we solve the inequality

$|\frac{1}{n}| < \varepsilon$ for n in terms of ε: $|1/n| < \varepsilon$ so $1/\varepsilon < n$ and $n > 1/\varepsilon$.

So for **any** positive ε, we can take $N = 1/\varepsilon$ (or any larger number).

Then $n > N = 1/\varepsilon$ implies that $\varepsilon > \frac{1}{n} = |\frac{1}{n}| = |\frac{n+1}{n} - 1| = |a_n - L|$.

Practice 7: $\lim_{n\to\infty} \frac{\ln(n)}{n}$: Since $\lim_{n\to\infty} \ln(x) = \infty$ and $\lim_{n\to\infty} x = \infty$, we can use L'Hopital's rule (Section 3.7). $\mathbf{D}(\ln(x)) = \frac{1}{x}$ and $\mathbf{D}(x) = 1$, so $\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{x\to\infty} \frac{1/x}{1} = 0$.

Then $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$ and $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$.

Practice 8: We can show that $a_n = \{ \sin(\frac{\pi n}{2}) \}$ diverges by finding two subsequences b_n and c_n of a_n so that the subsequences b_n and c_n converge to different limiting values.

Let $\{ b_n \}$ consist of the terms $\{ a_1, a_5, a_9, \ldots, a_{4n-3}, \ldots \} = \{ \sin(\pi/2), \sin(5\pi/2), \sin(9\pi/2), \ldots, \sin(\frac{(4n-3)\pi}{2}), \ldots \} = \{ 1, 1, 1, \ldots, 1, \ldots \}$. Then $\lim_{n\to\infty} b_n = 1$.

Let $\{ c_n \}$ consist of the terms $\{ a_2, a_4, a_6, \ldots, a_{2n}, \ldots \} = \{ \sin(2\pi/2), \sin(4\pi/2), \sin(6\pi/2), \ldots, \sin(\frac{2n\pi}{2}), \ldots \} = \{ 0, 0, 0, \ldots, 0, \ldots \}$. Then $\lim_{n\to\infty} c_n = 0$.

Since the subsequences $\{ b_n \}$ and $\{ c_n \}$ have different limits, we can conclude that the original sequence $\{ a_n \}$ diverges. (Note: Many other pairs of subsequences also work in place of the $\{ b_n \}$ and $\{ c_n \}$ that we used.)

Practice 9: (i) by showing that $a_{n+1} \le a_n$ for all n,

(ii) by showing that all the a_n are positive and $\frac{a_{n+1}}{a_n} \le 1$ for all n, or

(iii) by showing that $a_n = f(n)$ for integer values n and $f'(x) \le 0$ for all $x \ge 1$.

Practice 10: Show that $\left\{ (\frac{2}{3})^n \right\}$ is monotonically decreasing.

Using method (i) of Practice 9: $(\frac{2}{3}) < 1$ so multiplying each side by $(\frac{2}{3})^n > 0$ we have $(\frac{2}{3})(\frac{2}{3})^n < 1(\frac{2}{3})^n$ and $(\frac{2}{3})^{n+1} < (\frac{2}{3})^n$ so $a_{n+1} < a_n$.

We could have used method (ii) of Practice 9 instead: $a_{n+1} = (\frac{2}{3})^{n+1}$ and $a_n = (\frac{2}{3})^n$ so $\frac{a_{n+1}}{a_n} = \frac{(2/3)^{n+1}}{(2/3)^n} = \frac{2}{3} < 1$ so $\left\{ (\frac{2}{3})^n \right\}$ is monotonically decreasing.

Appendix: Proof of the Uniqueness Theorem

The proof starts by assuming that the limit is not unique. Then we show that this assumption of nonuniqueness leads to a contradiction so the assumption is false and the limit is unique.

Suppose that a sequence $\{ a_n \}$ converges to two different limits L_1 and L_2. Then the distance between L_1 and L_2 is $d = | L_1 - L_2 | > 0$. Since $\{ a_n \}$ converges to L_1, then for any $\varepsilon > 0$ there is an N_1 so that $n > N_1$ implies $| a_n - L_1 | < \varepsilon$. Take $\varepsilon = d/3$. Then there is an N_1 so $n > N_1$ implies $| a_n - L_1 | < \varepsilon = d/3$. Similarly, there is an N_2 so that $n > N_2$ implies $| a_n - L_2 | < \varepsilon = d/3$. Finally, if n is larger than both N_1 and N_2, then both conditions are satisfied and

$d = \| L_1 - L_2 \| = \| (a_n - L_2) - (a_n - L_1) \|$	by adding $0 = a_n - a_n$ inside the absolute value
$\le \| a_n - L_2 \| + \| a_n - L_1 \|$	by the Triangle Inequality
$< d/3 + d/3$	since $\| a_n - L_1 \| < \varepsilon = d/3$ and $\| a_n - L_2 \| < \varepsilon = d/3$
$= 2d/3$.	

We have found that $0 < d < \frac{2}{3} d$, a contradiction, and we can conclude the assumption, $L_1 \ne L_2$, is false. A sequence **can not** converge to two different values.

(Note: *We chose $\varepsilon = d/3$ because it "works" for our purpose by leading to a contradiction. Several other choices, any ε less than $d/2$, also lead to the contradiction. Since the definition says "for **any** $\varepsilon > 0$," we picked one we wanted.*)