10.10 TAYLOR AND MACLAURIN SERIES

This section discusses a method for representing a variety of functions as power series, and power series representations are derived for sin(x), cos(x), e^x, and several functions related to them. These power series are used to evaluate the functions and limits and to approximate definite integrals.

We start with an examination of how to determine the formula for a polynomial from information about the polynomial when $x = 0$, and then this process is extended to determine a series representation for a function from information about the function when $x = 0$.

Polynomials

Polynomials are among the easiest functions to work with, and they have a variety of "nice" properties including the following:

The values of P(x) and its derivatives at $x = 0$ completely determine the formula for P(x).

If the values of P(x) and all of its derivatives at $x = 0$ are known, then we can use those values to find a formula for P(x).

Example 1: Suppose P(x) is a cubic polynomial with $P(0) = 7$, $P'(0) = 5$, $P''(0) = 16$, and $P'''(0) = 18$. (Since P(x) is a cubic, its higher derivatives are all 0.) Find a formula for P(x).

Solution: Since P(x) is a cubic polynomial, then $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ for some numbers a_0, a_1, a_2, and a_3. We want to find the values of those numbers, and we can do so by substituting 0 for x in the expressions for P, P ', P '', and P ''' and using the given information.

$7 = P(0) = a_0 + a_1{\cdot}0 + a_2{\cdot}0^2 + a_3{\cdot}0^3 = a_0$ so $a_0 = 7$.

$P'(x) = a_1 + 2a_2x + 3a_3x^2$. $5 = P'(0) = a_1 + 2a_2{\cdot}0 + 3a_3{\cdot}0^2 = a_1$ so $a_1 = 5$.

$P''(x) = 2a_2 + 6a_3x$. $16 = P''(0) = 2a_2 + 6a_3{\cdot}0 = 2a_2$ so $a_2 = 16/2 = 8$.

$P'''(x) = 6a_3$. $18 = P''(0) = 6a_3$ so $a_3 = 18/6 = 3$.

$P(x) = 7 + 5x + 8x^2 + 3x^3$. You should verify that this cubic polynomial and its derivatives have the values specified in the problem.

Practice 1: Suppose P(x) is a 4th degree polynomial with $P(0) = -3$, $P'(0) = 4$, $P''(0) = 10$, $P'''(0) = 12$, and $P^{(4)}(0) = 24$. (Since P(x) is a 4th degree polynomial, the higher derivatives are all 0.) Find a formula for P(x).

For polynomials, the n_{th} derivative evaluated at $x = 0$ is $P^{(n)}(0) = (n)(n-1)(n-2)...(2)(1)\,a_n = n!{\cdot}a_n$, so the coefficient a_n of the n^{th} term of the polynomial is $a_n = P^{(n)}(0)/n!$.

Series

In many important ways power series behave like polynomials, very big polynomials, and this is one of those ways. The next result says that if a function can be represented by a power series, then the coefficients of the power series just depend on the values of the derivatives of the function evaluated at 0.

Maclaurin Series for f(x)

If a function $f(x)$ has a power series representation $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$

then the coefficients are given by $a_n = \frac{f^{(n)}(0)}{n!}$.

The **Maclaurin Series** for $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} \cdot x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \ldots + \frac{f^{(n)}(0)}{n!} \cdot x^n + \ldots$$

Proof: Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$ for $|x| < R$.

Then $f(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + \ldots + a_n \cdot 0^n + \ldots = a_0$ so $a_0 = f(0) = \frac{f^{(0)}(0)}{0!}$.

(We are using the conventions that $f^{(0)}(x) = f(x)$ and that $0! = 1$.)

$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots$

so $f'(0) = a_1 + 2a_2 \cdot 0 + 3a_3 \cdot 0^2 + \ldots + n a_n \cdot 0^{n-1} + \ldots = a_1$ and $a_1 = \frac{f'(0)}{1!}$.

$f''(x) = 2a_2 + 2 \cdot 3 a_3 x + \ldots + (n-1) \cdot n \cdot a_n x^{n-2} + \ldots$

so $f''(0) = 2a_2 + 2 \cdot 3 a_3 \cdot 0 + \ldots + (n-1) \cdot n \cdot a_n \cdot 0^{n-2} + \ldots = 2a_2$ and $a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$.

$f'''(x) = 2 \cdot 3 a_3 + \ldots + (n-2) \cdot (n-1) \cdot n \cdot a_n x^{n-3} + \ldots$

so $f'''(0) = 2 \cdot 3 a_3 + \ldots + (n-2) \cdot (n-1) \cdot n \cdot a_n \cdot 0^{n-3} + \ldots = 2 \cdot 3 a_3$ and $a_3 = \frac{f'''(0)}{2 \cdot 3} = \frac{f'''(0)}{3!}$.

In general, $f^{(n)}(x) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n\, a_n + \{\text{terms still containing powers of } x\}$

so $f^{(n)}(0) = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n\, a_n + \{ 0 \} = n!\, a_n$ and $a_n = \frac{f^{(n)}(0)}{n!}$.

A similar result, and proof, is also true for a "shifted" power series, a power series centered at some value c. Such shifted series are called Taylor series.

Taylor Series for f(x) centered at c

If a function f(x) has a power series representation $f(x) = \sum_{n=0}^{\infty} a_n(x - \mathbf{c})^n$ for $|x-c| < R$

then the coefficients are given by $a_n = \dfrac{f^{(n)}(\mathbf{c})}{n!}$.

The **Taylor Series** for f(x) at c is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\mathbf{c})}{n!}(x-\mathbf{c})^n = f(\mathbf{c}) + f'(\mathbf{c})\cdot(x-\mathbf{c}) + \frac{f''(\mathbf{c})}{2!}\cdot(x-\mathbf{c})^2 + \frac{f'''(\mathbf{c})}{3!}\cdot(x-\mathbf{c})^3 + \ldots + \frac{f^{(n)}(\mathbf{c})}{n!}\cdot(x-\mathbf{c})^n + \ldots$$

The proof is very similar to the proof for Maclaurin series and is not included here.

A Maclaurin series is a Taylor series centered at $c = 0$, and Maclaurin series are a special case of Taylor series.

Note: These statements for Maclaurin series and Taylor series do not say that every function is or can be written as a power series. However, if a function is a power series, then its coefficients must follow the given pattern. Fortunately, most of the important functions such as sin(x), cos(x), e^x, and ln(x) can be written as power series.

You should notice that the first term of the Taylor series is simply the value of the function f at the point $x = c$: it provides the best constant function approximation of f near $x = c$. The sum of the first two terms of the Taylor series pattern for a function, $f(c) + f'(c)\cdot(x-c)$, is the formula for the tangent line to f at $x = c$ and is the linear approximation of f(x) near $x = c$ that we first examined in Chapter 2. The Taylor series formula extends these approximations to higher degree polynomials, and the partial sums of the Taylor series provide higher degree polynomial approximations of f near $x = c$.

Taylor series and Maclaurin series were first discovered by the Scottish mathematician/astronomer James Gregory (1638–1675), but the results were not published until after his death. The English mathematician Brook Taylor (1685–1731) independently rediscovered these results and included them in a book in 1715. The Scottish mathematician/engineer Colin Maclaurin (1698–1746) quoted Taylor's work in his *Treatise on Fluxions* published in 1742. Maclaurin's book was widely read, and the Taylor series centered at $c = 0$ became known as Maclaurin series.

Example 2: Find the Maclaurin series for $f(x) = \sin(x)$ and the radius of convergence of the series.

Solution: $f(x) = \sin(x)$ so $f(0) = \sin(0)$ and $a_0 = f(0) = 0$.

$f'(x) = \cos(x)$ so $f'(0) = \cos(0) = 1$ and $a_1 = f'(0) = 1$.

$f''(x) = -\sin(x)$ so $f''(0) = -\sin(0) = 0$ and $a_2 = \frac{f''(0)}{2!} = 0$.

$f'''(x) = -\cos(x)$ so $f'''(0) = -\cos(0) = -1$ and $a_3 = \frac{f'''(0)}{3!} = \frac{-1}{3!}$.

$f^{(4)}(x) = \sin(x)$ and the pattern repeats:

$a_4 = 0, a_5 = \frac{1}{5!}, a_6 = 0, a_7 = \frac{-1}{7!}, a_8 = 0, a_9 = \frac{1}{9!}, \ldots$.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} .$$

Notice that the Maclaurin series for $\sin(x)$ alternates and contains only odd powers of x.

We use the Ratio Test to find the radius of convergence. Let $c_n = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. Then

$$c_{n+1} = (-1)^{n+1} \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} = (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!} \text{ so}$$

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{(-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!}}{(-1)^n \frac{x^{2n+1}}{(2n+1)!}} \right| = \left| \frac{x^{2n+3}}{x^{2n+1}} \frac{(2n+1)!}{(2n+3)!} \right| = \left| x^2 \frac{1}{(2n+2)(2n+3)} \right| \to 0 < 1$$

for every value of x.

The radius of convergence is $R = \infty$, and the interval of convergence is $(-\infty, \infty)$.

The Maclaurin series for $\sin(x)$ converges for every value of x.

Fig. 1 shows the graphs of $\sin(x)$ and the first few approximating polynomials x, $x - \frac{x^3}{3!}$, and $x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $-\pi \le x \le \pi$.

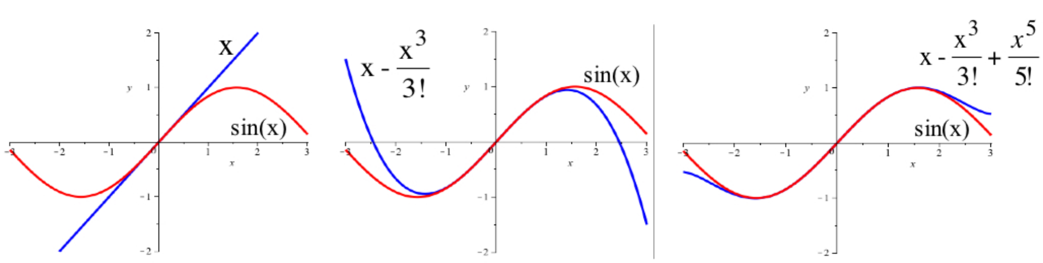

Fig. 1

By focusing our attention near $x = 0$, Fig. 1 shows the "goodness" of the Taylor polynomial fit to the function $f(x) = \sin(x)$. However, Fig. 2 shows that if x is not close to 0 then the values of the Taylor polynomials are far from the values of $f(x) = \sin(x)$. Typically the Taylor polynomials of a function are closest to the function when x is close to the number at which the series was centered, the value of c.

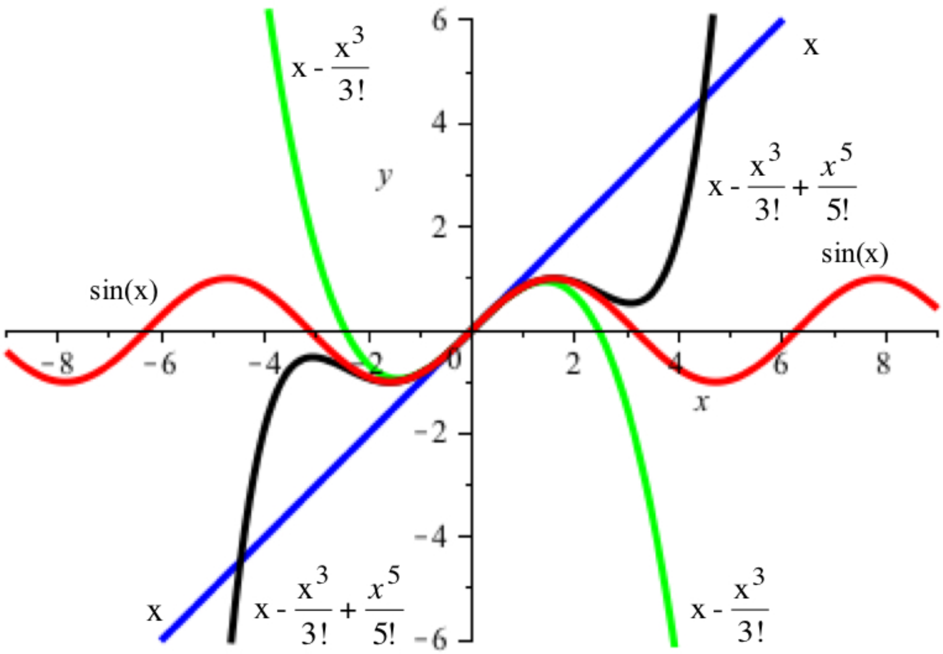

Fig. 2

Practice 2: Find the Maclaurin series for $f(x) = \cos(x)$ and the radius of convergence of the series. (Suggestion: Use the Term–by–Term Differentiation result of Section 10.9 .)

Once we have power series representations for $\sin(x)$ and $\cos(x)$, we can use the methods of Section 10.9 and the known series to approximate values of sine and cosine, determine power series representations of related functions, calculate limits, and approximate definite integrals.

Example 3: Use the series $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$ to represent $\sin(0.5)$ as a numerical series. Approximate the value of $\sin(0.5)$ by calculating the partial sum of the first three non–zero terms and give a bound on the "error" between this approximation and the exact value of $\sin(0.5)$.

Solution: $\sin(0.5) = (0.5) - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!} - \frac{(0.5)^7}{7!} + \frac{(0.5)^9}{9!} - \frac{(0.5)^{11}}{11!} + \ldots$.

$$\sin(0.5) \approx (0.5) - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!} = \frac{1}{2} - \frac{1}{48} + \frac{1}{3840} \approx 0.479427083333 .$$

Since this is an alternating series, the difference between the approximation and the exact value is less than the next term in the alternating series: "error" < $\frac{(0.5)^7}{7!} = \frac{1}{645120} \approx 0.00000155$.

If we use the sum of the first four nonzero terms to approximate the value of sin(0.5), then the "error" of the approximation is less than $\frac{(0.5)^9}{9!} = \frac{1}{185794560} \approx 5.4 \times 10^{-9}$.

We were able to obtain a bound for the error in the approximation of sin(0.5) because we were dealing with an alternating series, a type of series for which we have an error bound. However, many power series are not alternating series. In Section 10.11 we discuss a general error bound for Taylor series.

Practice 3: Use the sum of the first two nonzero terms of the Maclaurin series for cos(x) to approximate the value of cos(0.2). Give a bound on the "error" between this approximation and the exact value of cos(0.2).

Calculator Note: When you press the buttons on a calculator to evaluate sin(0.5) or cos(0.2), the calculator does not look up the answer in a table. Instead, the calculator is programmed with series representations for sine and cosine and other functions, and it calculates a partial sum of the appropriate series to obtain a numerical answer. It adds enough terms so that the 8 or 9 digits shown on the display are (usually) correct. In Section 10.11 we examine these methods in more detail and consider how to determine the number of terms needed in the partial sum to achieve the desired number of accurate digits in the answer.

Example 4: Represent $\sin(x^3)$ and $\int \sin(x^3)\,dx$ as power series. Then use the first three nonzero terms to approximate the value of $\int_0^1 \sin(x^3)\,dx$ and obtain a bound on the "error" of this approximation.

Solution: $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$ so

$$\sin(x^3) = (x^3) - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \frac{(x^3)^7}{7!} \ldots = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} \ldots .$$

$$\int_0^1 \sin(x^3)\,dx = \int_0^1 \left\{ x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} \ldots \right\} dx = \frac{x^4}{4} - \frac{x^{10}}{10 \cdot 3!} + \frac{x^{16}}{16 \cdot 5!} - \frac{x^{22}}{22 \cdot 7!} + \ldots \Big|_0^1$$

$$= \left\{ \frac{1}{4} - \frac{1}{10 \cdot 3!} + \frac{1}{16 \cdot 5!} - \frac{1}{22 \cdot 7!} + \ldots \right\} - \{ 0 \} .$$

$\frac{1}{4} - \frac{1}{10 \cdot 3!} + \frac{1}{16 \cdot 5!} \approx 0.2338542$ and $\frac{1}{22 \cdot 7!} = \frac{1}{110880} \approx 0.0000090$ so

$\int_0^1 \sin(x^3)\, dx \approx 0.2338542$ and this approximation is within 0.0000090 of the exact value.

If we took just one more term, $\int_0^1 \sin(x^3)\, dx \approx \frac{1}{4} - \frac{1}{10 \cdot 3!} + \frac{1}{16 \cdot 5!} - \frac{1}{22 \cdot 7!} \approx 0.233845515$ is within $\frac{1}{28 \cdot 9!} \approx 0.000000098$ of the exact value of the integral.

Practice 4: Represent $x \cdot \cos(x^3)$ and $\int x \cdot \cos(x^3)\, dx$ as power series. Then use the first two nonzero terms to approximate the value of $\int_0^{1/2} x \cdot \cos(x^3)\, dx$ and obtain a bound on the "error" of this approximation.

Graphically

Each partial sum of the series $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$ contains a finite number of terms and is simply a polynomial:

$P_1(x) = x$

$P_3(x) = x - \frac{x^3}{3!} = x - \frac{1}{6} x^3$

$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} = x - \frac{1}{6} x^3 + \frac{1}{120} x^5$, ...

Fig. 1 showed the graphs of $\sin(x)$ and $P_1(x), P_3(x)$, and $P_5(x)$. As you saw, all of these polynomials are "good" approximations of $\sin(x)$ when x is very close to 0. The higher degree polynomials $P_n(x)$ provide "good" approximations of $\sin(x)$ over larger intervals.

Power Series for e^x

Example 5: Find the Maclaurin series for $f(x) = e^x$ and the radius of convergence of the series.

Solution: This is a very important series.

$f(x) = e^x$ so $f(0) = e^0 = 1$ and $a_0 = f(0) = 1$.

$f'(x) = e^x$ so $f'(0) = e^0 = 1$ and $a_1 = f'(0) = 1$.

$f''(x) = e^x$ so $f''(0) = e^0 = 1$ and $a_2 = \frac{f''(0)}{2!} = \frac{1}{2!}$.

For every value of n, $f^{(n)}(x) = e^x$ so $f^{(n)}(0) = e^0 = 1$ and $a_n = \frac{f''(0)}{n!} = \frac{1}{n!}$. Then

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

We can use the Ratio Test to find the radius of convergence. $c_n = \frac{x^n}{n!}$ so $c_{n+1} = \frac{x^{n+1}}{(n+1)!}$.

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \left| \frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!} \right| = \left| x \cdot \frac{1}{n+1} \right| \rightarrow 0 < 1 \text{ for every value of } x.$$

The radius of convergence is $R = \infty$, and the interval of convergence is $(-\infty, \infty)$. The Maclaurin series for e^x converges for every value of x.

Practice 5: Evaluate the partial sums of the first six terms of the numerical series for $e = e^1$ and $\frac{1}{\sqrt{e}} = e^{-1/2}$ and compare these partial sums with the values your calculator gives.

(Note: The numerical series for e^1 is not an alternating series so we do not have a bound for the approximation yet. We will in the next section.)

Fig. 3 shows the graphs of e^x and the approximating polynomials $1 + x$, $1 + x + \frac{x^2}{2!}$, and $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ for values of near 0.

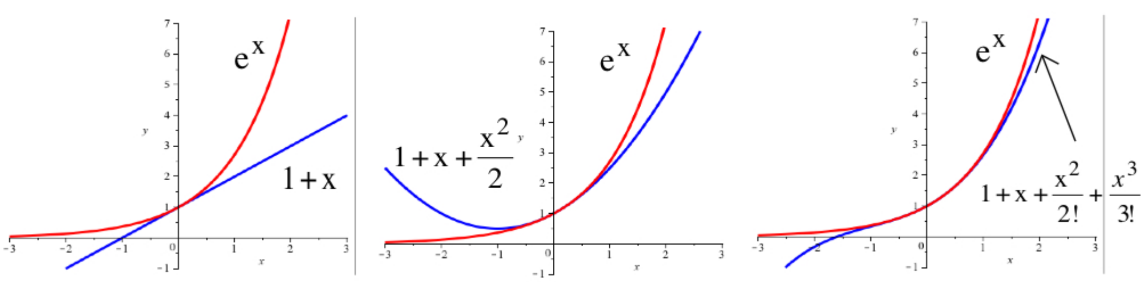

Fig. 3

The following series converge for all values of x in the interval I:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad I = (-\infty, \infty).$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad I = (-\infty, \infty).$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad I = (-\infty, \infty).$$

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} \qquad I = (0, 2].$$

Typically, these series converge very quickly to the value of the functions when x is close to 0 (or when x is close to 1 for ln(x)), but the convergence can be rather slow when x is far from 0. For example, the first 2 terms of the Taylor series for sine approximate sin(0.1) correctly to 6 decimal places, but 11 terms are needed to approximate sin(5) with the same accuracy.

Multiplying Power Series

We can add and subtract power series term–by–term, and we have already multiplied power series by single terms such as x and x^2, but occasionally it is useful to multiply a power series by another power series. The method for multiplying series is the same method we use to multiply a polynomial by another polynomial, but it becomes very tedious to get more than the first few terms of the resulting product.

Example 6: Find the first 5 nonzero terms of

$$\frac{1}{1-x} \cdot \sin(x) = (1 + x + x^2 + x^3 + \ldots) \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right).$$

Solution:

$$\begin{array}{lccccccl}
 & 1 & +x & +x^2 & +x^3 & +x^4 & +x^5 & + \ldots \\
\text{times} & & x & & -\frac{x^3}{6} & & +\frac{x^5}{120} & - \ldots \\
\hline
 & & x & +x^2 & +x^3 & +x^4 & +x^5 & + \ldots \quad \text{(from multiplying by x)} \\
 & & & & -\frac{1}{6}x^3 & -\frac{1}{6}x^4 & -\frac{1}{6}x^5 & + \ldots \quad \text{(from multiplying by } -x^3/6\text{)} \\
 & & & & & & +\frac{1}{120}x^5 & + \ldots \quad \text{(from multiplying by } x^5/120\text{)} \\
\hline
= & & x & +x^2 & +\frac{5}{6}x^3 & +\frac{5}{6}x^4 & +\frac{101}{120}x^5 & + \ldots \quad \text{(from adding previous terms)}
\end{array}$$

Practice 6: Find the first 3 nonzero terms of

$$e^x \cdot \sin(x) = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots\right) \cdot \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right).$$

It is also possible to divide one power series by another power series using a procedure similar to "long division" of a polynomial by a polynomial, but we will not discuss that algorithm.

PROBLEMS

1. Find a formula for a polynomial P of degree 2 such that $P(0) = 7$, $P'(0) = -5$, and $P''(0) = 8$.
2. Find a formula for a polynomial P of degree 3 such that $P(0) = -1$, $P'(0) = 2$, $P''(0) = -5$, and $P'''(0) = 12$.
3. Find a formula for a polynomial P of degree 2 such that $P(3) = -2$, $P'(3) = 5$, and $P''(3) = 4$.
4. Find a formula for a polynomial P of degree 2 such that $P(1) = -2$, $P'(1) = 5$, and $P''(1) = 4$.

In problems 5 – 8, calculate the first several terms of the Maclaurin series for the given functions and compare with the series representations we found in Section 10.9 . (The series should be the same.)

5. $\ln(1 + x)$ to the x^4 term
6. $\ln(1 - x)$ to the x^4 term
7. $\arctan(x)$ to the x^3 term
8. $1/(1 - x)$ to the x^4 term

In problems 9 – 12, calculate the first several terms of the Maclaurin series for the given functions.

9. $\cos(x)$ to the x^6 term
10. $\tan(x)$ to the x^5 term
11. $\sec(x)$ to the x^4 term
12. e^{3x} to the x^4 term

In problems 13 - 18, calculate the first several terms of the Taylor series for the given functions at the given point c.

13. $\ln(x)$ for $c = 1$

14. $\sin(x)$ for $c = \pi$

15. $\sin(x)$ for $c = \pi/2$

16. $\sqrt{x}$ for $c = 1$

17. $\sqrt{x}$ for $c = 9$

In problems 18 – 21, use the first three nonzero terms of the Maclaurin series for each function to approximate the numerical values. Then compare the Maclaurin series approximation with the value your calculator gives.

18. $\sin(0.1)$, $\sin(0.2)$, $\sin(0.5)$, $\sin(1)$, and $\sin(2)$

19. $\cos(0.1)$, $\cos(0.2)$, $\cos(0.5)$, $\cos(1)$, and $\cos(2)$

20. $\ln(1.1)$, $\ln(1.2)$, $\ln(1.3)$, $\ln(2)$, and $\ln(3)$

21. $\arctan(0.1)$, $\arctan(0.2)$, $\arctan(0.5)$, $\arctan(1)$, and $\arctan(2)$

In problems 22 – 27, calculate the first three nonzero terms of the power series for each of the integrals.

22. $\int \cos(x^2)\,dx$ and $\int \cos(x^3)\,dx$

23. $\int \sin(x^2)\,dx$ and $\int \sin(x^3)\,dx$

24. $\int e^{(x^2)}\,dx$ and $\int e^{(x^3)}\,dx$

25. $\int e^{(-x^2)}\,dx$ and $\int e^{(-x^3)}\,dx$

26. $\int \ln(x)\,dx$ and $\int x\ln(x)\,dx$

27. $\int x\sin(x)\,dx$ and $\int x^2\sin(x)\,dx$

In problems 28 – 35, use the series representation of these functions to calculate the limits.

28. $\lim_{x\to 0} \frac{1-\cos(x)}{x}$

29. $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$

30. $\lim_{x\to 0} \frac{\ln(x)}{x-1}$

31. $\lim_{x\to 0} \frac{1-e^x}{x}$

32. $\lim_{x\to 0} \frac{1+x-e^x}{x^2}$

33. $\lim_{x\to 0} \frac{\sin(x)}{x}$

34. $\lim_{x\to 0} \frac{x-\sin(x)}{x^3}$

35. $\lim_{x\to 0} \frac{x-\frac{x^3}{6}-\sin(x)}{x^5}$

36. Use the series for e^x and e^{-x} to write a series representation for $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.
(The function "cosh" is called the hyperbolic cosine function.)

37. Use the series for e^x and e^{-x} to write a series representation for $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$.
(The function "sinh" is called the hyperbolic sine function.)

38. Show that **D(** series for cosh(x) in problem 36 **)** is the series for sinh(x) in problem 37.

39. Show that **D(** series for sinh(x) in problem 37 **)** is the series for cosh(x) in problem 36.

Euler's Formula

So far we have only discussed series with real numbers, but sometimes it is useful to replace the variable with complex numbers. The next problems ask you to make such a substitution and then to derive and use one of the most famous formulas in mathematics, Euler's formula. (Recall that $i = \sqrt{-1}$ is called the complex unit and that its powers follow the pattern $i^2 = -1$, $i^3 = (i^2)(i) = -i$, $i^4 = (i^2)(i^2) = 1$, $i^5 = (i^4)(i) = i$, ...)

40. Start with the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \ldots$
   (a) Substitute "ix" for "x" and write a series for e^{ix}.
   (b) In part (a) simplify each power of i and write a simplified series for e^{ix}. (e.g., $(ix)^3/3!$ simplifies to $i^3x^3/3! = -i \cdot x^3/3!$)
   (c) Sort the terms in the series in part (b) into those terms that do not contain i and those terms that do contain i. Then rewrite the series for e^{ix} in the form
   $e^{ix} = \{$ terms that did not contain i $\} + i \cdot \{$ terms that did contain i $\}$.
   (d) You should recognize the sum in each bracket in part (c) as the series for an elementary function (hint: think trigonometry). Rewrite the pattern in part (c) as
   $e^{ix} = \{$ function $\} + i \cdot \{$ another function $\}$.

41. The answer you should have gotten in problem 40d, $\mathbf{e^{ix} = \cos(x) + i \cdot \sin(x)}$, is called Euler's formula. Use Euler's formula to calculate the values of $e^{i(\pi/2)}$ and $e^{\pi i}$.

42. Use Euler's formula to show that $e^{\pi i} + 1 = 0$. This is one of the most remarkable formulas in mathematics because it connects five of the most fundamental constants (the additive identity 0, the multiplicative identity 1, the complex unit i, and the two most commonly used irrational numbers π and e in a simple but non-obvious way.

Binomial Series

You have probably seen the pattern for expanding $(1 + x)^n$ where n is a nonnegative integer:

$$(1 + x)^0 = 1$$
$$(1 + x)^1 = 1 + x$$
$$(1 + x)^2 = 1 + 2x + x^2$$
$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$$
$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$
$$(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

Row	Pascal's Triangle
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1

Each number in Pascal's Triangle is the sum of the two numbers closest to it in the row immediately above it.

Fig. 4

either using Pascal's triangle (Fig. 4) or using the binomial coefficients, written $\binom{n}{k}$ and defined as

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!} .$$

43. Calculate the binomial coefficients $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$ and verify that
    (i) they agree with the entries in the 3rd row of Pascal's triangle
    (ii) they agree with the coefficients of the terms of $(1 + x)^3$.

44. Calculate the binomial coefficients $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, and $\binom{4}{4}$ and verify that
    (i) they agree with the entries in the 4th row of Pascal's triangle
    (ii) they agree with the coefficients of the terms of $(1 + x)^4$.

Using binomial coefficients, the pattern for nonnegative integer powers of $(1 + x)$ can be described in a very compact way:

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k .$$

When n is a positive integer, $(1 + x)^n$ expands to be a polynomial of degree n.
But what happens when n is a negative integer or perhaps not even an integer? This was a question that Newton himself investigated, and it led him to a general pattern, called the Binomial Series Theorem, for $(1 + x)^m$ when m is any real number. And now you can do it, too.

45. Let $f(x) = (1 + x)^{5/2}$ and determine the first 5 terms of the Maclaurin series for f(x).

46. Let $f(x) = (1 + x)^{-3/2}$ and determine the first 5 terms of the Maclaurin series for f(x).

47. Let $f(x) = (1 + x)^m$ and determine the first 4 terms of the Maclaurin series for $f(x)$. This is the start of the derivation of the Binomial Series Theorem given below.

Binomial Series Theorem

If m is any real number and $|x| < 1$

then $$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \ldots = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

where $\binom{m}{k} = \frac{m(m-1)(m-2) \cdots (m-k+1)}{k!}$ (for $k \geq 1$) and $\binom{m}{0} = 1$.

48. Use the Ratio Test to show that $\sum_{k=0}^{\infty} \binom{m}{k} x^k$ converges for $|x| < 1$.

Practice Answers

Practice 1: $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ with $P(0) = -3$, $P'(0) = 4$, $P''(0) = 10$, $P'''(0) = 12$, and $P^{(4)}(0) = 24$.

$-3 = P(0) = a_0 + a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 0 + a_4 \cdot 0 = a_0$ so $a_0 = -3$

$P'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$

$4 = P'(0) = a_1 + 2a_2 \cdot 0 + 3a_3 \cdot 0 + 4a_4 \cdot 0 = a_1$ so $a_1 = 4$

$P''(x) = 2a_2 + 6a_3x + 12a_4x^2$

$10 = P''(0) = 2a_2 + 6a_3 \cdot 0 + 12a_4 \cdot 0 = 2a_2$ so $a_2 = 10/2 = 5$

$P'''(x) = 6a_3 + 24a_4x$

$12 = P'''(0) = 6a_3 + 24a_4 \cdot 0 = 6a_3$ so $a_3 = 12/6 = 2$

$P^{(4)}(x) = 24a_4$

$24 = P^{(4)}(0) = 24a_4$ so $a_4 = 24/24 = 1$.

Then $P(x) = -3 + 4x + 5x^2 + 2x^3 + 1x^4$.

Practice 2: $\cos(x) = \mathbf{D}(\sin(x)) = \mathbf{D}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots\right)$

$$= 1 - 3\cdot\frac{x^2}{3!} + 5\cdot\frac{x^4}{5!} - 7\cdot\frac{x^6}{7!} + 9\cdot\frac{x^8}{9!} - 11\cdot\frac{x^{10}}{11!} + \ldots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Practice 3: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$

Using the first two nonzero terms, $\cos(0.2) \approx 1 - \frac{(0.2)^2}{2!} = 1 - \frac{0.04}{2} = 0.98$.

Since $\cos(0.2) = 1 - \frac{(0.2)^2}{2!} + \frac{(0.2)^4}{4!} - \frac{(0.2)^6}{6!} + \ldots$ is a convergent alternating series, the error is less than the absolute value of the next term.

Then $\cos(0.2) \approx 1 - \frac{(0.2)^2}{2!} = 0.98$ with an error less than $\left| \frac{(0.2)^4}{4!} \right| = \frac{0.0016}{24} \approx 0.000067$: $|\cos(0.2) - 0.98| < 0.000067$. (In fact, $\cos(0.2) \approx 0.9800665778$.)

Practice 4: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$

$$\cos(x^3) = 1 - \frac{(x^3)^2}{2!} + \frac{(x^3)^4}{4!} - \frac{(x^3)^6}{6!} + \frac{(x^3)^8}{8!} - \ldots = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \frac{x^{24}}{8!} - \frac{x^{30}}{10!} + \ldots$$

$$x\cdot\cos(x^3) = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \frac{x^{25}}{8!} - \frac{x^{31}}{10!} + \ldots$$

$$\int x\cdot\cos(x^3)\, dx = \frac{x^2}{2} - \frac{1}{8}\cdot\frac{x^8}{2!} + \frac{1}{14}\cdot\frac{x^{14}}{4!} - \frac{1}{20}\cdot\frac{x^{20}}{6!} + \frac{1}{26}\cdot\frac{x^{26}}{8!} - \frac{1}{32}\cdot\frac{x^{32}}{10!} + \ldots + C$$

$$\int_0^{1/2} x\cdot\cos(x^3)\, dx \approx \left. \frac{x^2}{2} - \frac{1}{8}\cdot\frac{x^8}{2!} \right|_0^{1/2} = \left(\frac{(0.5)^2}{2} - \frac{(0.5)^8}{2!8} \right) - (0) \approx 0.124755859$$

with $|\text{error}| \le \frac{(0.5)^{14}}{4!14} \approx 1.82\cdot 10^{-7} = 0.000000182$.

Practice 5: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$. Using the first six terms,

$e^1 \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} \approx 2.71666666666$ (My calculator gives $e^1 \approx 2.718281828$)

$$e^{-1/2} \approx 1 + (-1/2) + \frac{(-1/2)^2}{2!} + \frac{(-1/2)^3}{3!} + \frac{(-1/2)^4}{4!} + \frac{(-1/2)^5}{5!} \approx 0.6065104167$$

(My calculator gives $e^{-1/2} \approx 0.6065306597$).

Practice 6: $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$ $(= e^x)$

times $x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$ $(= \sin(x))$

$x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \ldots$ (from multiplying by x)

$-\frac{x^3}{6} - \frac{x^4}{6} - \frac{x^5}{12} - \ldots$ (from multiplying by $-x^3/6$)

$\frac{x^5}{120} + \ldots$ (from multiplying by $x^5/120$)

product is $x + x^2 + \frac{x^3}{3} + 0 - \frac{-9x^5}{120} + \ldots$ (from adding the previous terms)

The sum of the first three nonzero terms is $e^x \cdot \sin(x) = x + x^2 + \frac{x^3}{3} + \ldots$.

MAPLE command to plot Fig. 2

```
plot({x, x-x^3/6, x-x^3/6+x^5/120, sin(x)}, x=-9..9, y=-6..6, color=[blue, green, black, red], thickness=3);
```