14.2 Limits and Continuity of Multivariable Functions

Just as with a function of a single variable, before we can investigate differentiation we must consider limits and continuity. Although limits of functions of two (or more) variables present some additional complications, for the most part the definitions and properties of limits for these functions closely resemble the definitions and properties given in Chapter 1 for functions of a single variable.

You may want to spend some time now reviewing Chapter 1 to refamiliarize yourself with the main ideas and results for limits of functions of one variable—and rework a few limit problems.

Limits of Functions of Two Variables

One intuitive interpretation of the statement $\lim_{x \to a} f(x) = L$ was that "we can guarantee that $f(x)$ is as close to L as we like by restricting x to be sufficiently close (but not necessarily equal) to a." The precise definition of this concept said:

> Given any $\epsilon > 0$ there exists a $\delta > 0$ so that:
>
> $$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

When working with one variable, x could approach a in one of two ways: from the left or from the right. Accordingly, we defined left ($x \to a^-$) and right ($x \to a^+$) limits and determined that when both of these limits existed and were equal to the number L, the two-sided limit existed and was equal to L. With functions of two variables, you can approach a point (x, y) not only from an infinite number of directions, but along an even greater infinite number of paths.

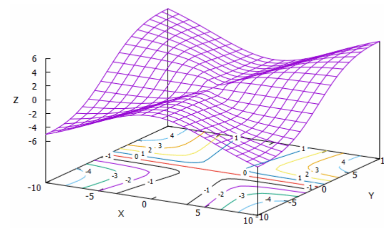
Example 1. Investigate the behavior of $f(x,y) = \dfrac{x^2 \cdot y}{x^2 + y^2}$ near $(0,0)$.

Solution. Note first that $f(x,y)$ is undefined at $(0,0)$, as its formula takes on the form "$\frac{0}{0}$"; this is not an issue if you are only concerned about the behavior of the function *near* $(0,0)$. Constructing a table:

y \ x	-0.10	-0.08	-0.06	-0.04	-0.02	0.00	0.02	0.04	0.06	0.08	0.10
0.10	0.050	0.039	0.026	0.014	0.004	0.000	0.004	0.014	0.026	0.039	0.050
0.08	0.049	0.040	0.029	0.016	0.005	0.000	0.005	0.016	0.029	0.040	0.049
0.06	0.044	0.038	0.030	0.018	0.006	0.000	0.006	0.018	0.030	0.038	0.044
0.04	0.034	0.032	0.028	0.020	0.008	0.000	0.008	0.020	0.028	0.032	0.034
0.02	0.019	0.019	0.018	0.016	0.010	0.000	0.010	0.016	0.018	0.019	0.019
0.00	0.000	0.000	0.000	0.000	0.000	**UND**	0.000	0.000	0.000	0.000	0.000
-0.02	-0.019	-0.019	-0.018	-0.016	-0.010	0.000	-0.010	-0.016	-0.018	-0.019	-0.019
-0.04	-0.034	-0.032	-0.028	-0.020	-0.008	0.000	-0.008	-0.020	-0.028	-0.032	-0.034
-0.06	-0.044	-0.038	-0.030	-0.018	-0.006	0.000	-0.006	-0.018	-0.030	-0.038	-0.044
-0.08	-0.049	-0.040	-0.029	-0.016	-0.005	0.000	-0.005	-0.016	-0.029	-0.040	-0.049
-0.10	-0.050	-0.039	-0.026	-0.014	-0.004	0.000	-0.004	-0.014	-0.026	-0.039	-0.050

indicates that the values of $f(x,y)$ generally appear to be getting closer to 0 as the inputs (x,y) get nearer to $(0,0)$.

A graph of this function (see margin) also seems to indicate the values of $f(x,y)$ are all reasonably close to 0 for inputs near $(0,0)$. Unfortunately, neither a table nor a graph *proves* that near $(0,0)$ all of the function values are close to 0, but using some algebra:



$$0 \leq \left| \frac{x^2 \cdot y}{x^2+y^2} \right| = |y| \cdot \frac{x^2}{x^2+y^2} \leq |y| \cdot \frac{x^2+y^2}{x^2+y^2} = |y|$$

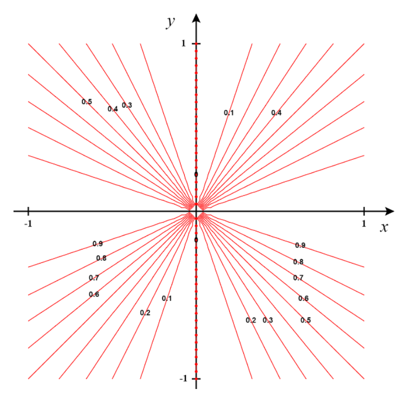
As (x,y) approaches $(0,0)$ along any path whatsoever, $y \to 0$, so the inequality above guarantees that $f(x,y) \to 0$ as well. ◀

Example 2. Investigate the behavior of $g(x,y) = \dfrac{x^2}{x^2+y^2}$ near $(0,0)$.

Solution. Again, $g(x,y)$ is undefined at $(0,0)$, but this is not an issue if our concern is what happens *near* $(0,0)$. If (x,y) approaches $(0,0)$ along the x-axis (where $y=0$) we have:

$$\frac{x^2}{x^2+y^2} = \frac{x^2}{x^2+0^2} = \frac{x^2}{x^2} = 1$$

for any $x \neq 0$. Along the y-axis (where $x=0$), however:



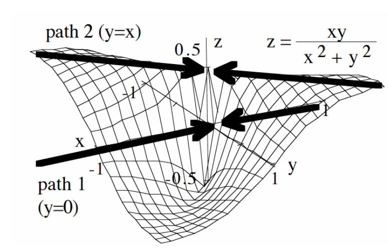
$$\frac{x^2}{x^2+y^2} = \frac{0^2}{0^2+y^2} = \frac{0}{y^2} = 0$$

for $y \neq 0$. Because $g(x,y) = 1$ at infinitely many points arbitrarily close to $(0,0)$ and $g(x,y) = 0$ at infinitely many other points arbitrarily close to $(0,0)$, $g(x,y)$ has no single limiting value as $(x,y) \to (0,0)$. A contour map for $g(x,y)$ (see margin) shows level curves for many different levels all approaching $(0,0)$. (What does this tell you?) ◀

Practice 1. Investigate the behavior of $F(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ near $(0,0)$.

Example 3. Investigate the behavior of $G(x,y) = \dfrac{xy}{x^2+y^2}$ near $(0,0)$.

Solution. On the x-axis, $G(x,0) = 0$ for $x \neq 0$. Similarly, on the y-axis, $G(0,y) = 0$ for $y \neq 0$, so 0 seems to be a good candidate for a limit. Unfortunately, along the line $y = x$ (see margin figure for a graph):



$$G(x,x) = \frac{x \cdot x}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

for $x \neq 0$, so $G(x,y)$ has no single limiting value as $(x,y) \to (0,0)$. ◀

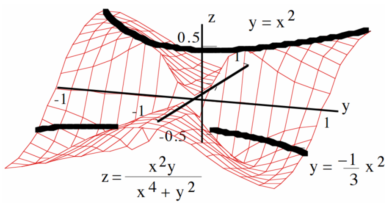
Practice 2. Investigate the behavior of $G(x,y) = \dfrac{xy}{x^2+y^2}$ along the line $y = mx$ for an arbitrary slope m.

Example 4. Investigate the behavior of $f(x,y) = \dfrac{x^2 \cdot y}{x^4+y^2}$ near $(0,0)$.

Solution. On the x-axis, $f(x,0) = 0$ for $x \neq 0$; on the y-axis, $f(0,y) = 0$ for $y \neq 0$. Along a generic line through the origin $y = mx$:

$$f(x,mx) = \frac{x^2 \cdot mx}{x^4 + (mx)^2} = \frac{mx^3}{x^2\left(x^2+m^2\right)} = \frac{mx}{x^2+m^2} \to 0$$

as $x \to 0$, so $f(x,y)$ approaches 0 along any straight path approaching $(0,0)$. The simplest non-linear curve is a parabola, and along $y = x^2$:



$$f(x,x^2) = \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \frac{x^4}{2x^4} = \frac{1}{2}$$

Foiled again! (See margin figure for a graph.) ◀

Practice 3. Investigate the behavior of $f(x,y) = \dfrac{x^2 \cdot y}{x^4+y^2}$ along a generic parabola passing through the origin of the form $y = Ax^2$.

Clearly you cannot check *every* possible path to a limiting input when attempting to determine a limiting value of a function, but the preceding Examples help show that we need a precise definition of a limit for this very reason.

Precise Definition of a Limit

The vector notation introduced at the end of Section 14.1 allows us to state a formal definition of a limit for a function of two variables that looks remarkably similar to the definition stated in Section 1.4 for functions of a single variable.

> **Definition of a Limit**: If, given any $\epsilon > 0$, there exists a $\delta > 0$ so that:
>
> $$0 < \|\mathbf{x} - \mathbf{a}\| < \delta \quad \Rightarrow \quad |f(\mathbf{x}) - L| < \epsilon$$
>
> we write $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ and say that "the limit of $f(\mathbf{x})$, as $\mathbf{x}$ approaches $\mathbf{a}$, equals L."

Example 5. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$.

Solution. From Example 1 we know that:

$$0 \leq \left|\frac{x^2 \cdot y}{x^2+y^2}\right| = |y| \cdot \frac{x^2}{x^2+y^2} \leq |y| \cdot \frac{x^2+y^2}{x^2+y^2} = |y|$$

and furthermore we know that:

$$|y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2} = \| \langle x, y \rangle - \langle 0, 0 \rangle \|$$

Given any number $\epsilon > 0$, let $\delta = \epsilon$. Then:

$$0 < \| \langle x, y \rangle - \langle 0, 0 \rangle \| < \delta = \epsilon \;\Rightarrow\; \left| \frac{x^2 \cdot y}{x^2 + y^2} - 0 \right| \leq |y| < \epsilon$$

which is what we need to show, according to the limit definition. ◀

Example 6. Show that $\displaystyle\lim_{(x,y)\to(a,b)} x = a$ and $\displaystyle\lim_{(x,y)\to(a,b)} y = b$.

Solution. Given $\epsilon > 0$, let $\delta = \epsilon$ and note that:

$$|x - a| = \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2 + (y-b)^2} = \| \langle x, y \rangle - \langle a, b \rangle \|$$

so that:

$$0 < \| \langle x, y \rangle - \langle a, b \rangle \| < \delta = \epsilon \quad \Rightarrow \quad |x - a| < \epsilon$$

as required. The other (very similar) result is left for you to prove. ◀

As you might suspect, the results of the Main Limit Theorem from Section 1.2 carry over to limits of functions of two (or more) variables (with very similar proofs, which we will omit here).

Main Limit Theorem:

If $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M$

then

(a) $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} [f(\mathbf{x}) + g(\mathbf{x})] = L + M$

(b) $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} [f(\mathbf{x}) - g(\mathbf{x})] = L - M$

(c) $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} k \cdot f(\mathbf{x}) = k \cdot L$

(d) $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) \cdot g(\mathbf{x}) = L \cdot M$

(e) $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{L}{M}$ (if $M \neq 0$)

(f) $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} [f(\mathbf{x})]^n = L^n$

(g) $\displaystyle\lim_{\mathbf{x}\to\mathbf{a}} \sqrt[n]{f(\mathbf{x})} = \sqrt[n]{L}$

When n is an even integer in part (g) of the Main Limit Theorem, we need $L \geq 0$ and $f(\mathbf{x}) \geq 0$ for $\mathbf{x}$ near $\mathbf{a}$.

Combining the results of Example 6 with the Main Limit Theorem allows us to compute a wide variety of limits very quickly.

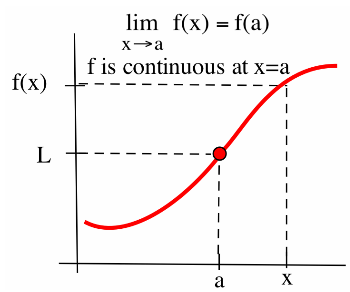
Example 7. Compute $\displaystyle\lim_{(x,y)\to(3,4)} 5x^2\sqrt{y}$.

Solution. Using results from the Main Limit Theorem and Example 6:

$$\lim_{(x,y)\to(3,4)} 5x^2\sqrt{y} = 5\cdot\left[\lim_{(x,y)\to(3,4)} x\right]^2 \cdot \sqrt{\lim_{(x,y)\to(3,4)} y} = 5\cdot 3^2\cdot\sqrt{4} = 90$$

which agrees with substituting $x = 3$ and $y = 4$ into the expression. ◀

Practice 4. Compute $\displaystyle\lim_{(x,y)\to(2,-5)} \frac{3x^2+6xy-7y^2+12}{(x+2)^2+(y-5)^2}$.

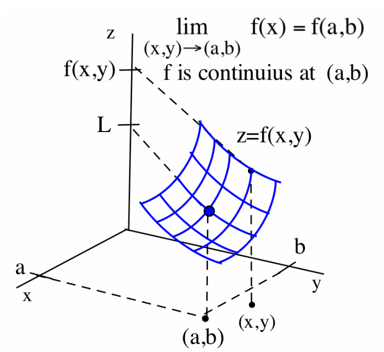


Continuity

A function of one variable is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. Graphically, this means that the graph of f is "connected" at the point $(a, f(a))$ and does not have a hole or break there (see margin). The definition and meaning of continuity for functions of two (or more) variables is quite similar.

Definition of Continuity: A function f of two (or more) variables defined at **a** and for all points near **a** is **continuous** at **a** if:

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$



Graphically, this definition means that the graph of f does not have a hole or break at any point of continuity (see margin).

Just as we talked about a function of one variable being continuous on an interval (or the entire real number line), we can talk about a function of n variables being continuous on a region $\mathcal{R}$ in $\mathbb{R}^n$.

Definition: A function f is continuous on a region $\mathcal{R}$ if it is continuous at each point in $\mathcal{R}$.

Most of the functions of two (or more) variables you will encounter are continuous either everywhere (for example, at all points (x, y) in the plane for a function of two variables) or continuous everywhere except at a "few" places. The results of Example 6 and the Main Limit Theorem allow us to show quite easily that any polynomial function, such as $f(x,y) = 3x^3 - 5x^2y + 7xy^2 + 9y^4 - 17$, is continuous everywhere, and that any rational function is continuous everywhere except at points where its denominator equals 0.

Example 8. Where is $f(x,y) = \dfrac{x^2y}{x^2+y^2}$ continuous?

Solution. Because $f(x, y)$ is a rational function, it is continuous everywhere except where the denominator is 0, and $x^2 + y^2 = 0$ only at $(0, 0)$. Because $f(x, y)$ is undefined at $(0, 0)$, it is not continuous there. ◀

Practice 5. Where is $g(x,y) = \dfrac{3x^2 + 6xy - 7y^2 + 12}{(x+2)^2 + (y-5)^2}$ continuous?

Example 9. Define $F(x,y) = \dfrac{x^2 y}{x^2 + y^2}$ for $(x,y) \neq (0,0)$ with $F(0,0) = 0$. Where is $F(x,y)$ continuous?

Solution. At any point other than $(0,0)$, $F(x,y)$ is continuous because it is a rational function with nonzero denominator. At $(0,0)$, we know from Example 5 that $\lim_{(x,y)\to(0,0)} F(x,y) = 0 = F(0,0)$, so $F(x,y)$ is continuous there as well, hence on all of $\mathbb{R}^2$. ◀

Practice 6. Define $G(x,y) = \dfrac{x^2 y}{x^2 + y^2}$ for $(x,y) \neq (0,0)$ with $G(0,0) = 1$. Where is $G(x,y)$ continuous?

As with functions of a single variable, the Main Limit Theorem allows us to conclude that sums, differences, products and quotients of continuous functions are also continuous. Furthermore, the result from Section 1.3 about continuity of compositions of functions also extends quite easily (with a similar proof) to situations where the "inner" function is a function of two (or more) variables.

For quotients of continuous functions, the usual caveat about the denominator being nonzero applies.

Composition of Continuous Functions:

If $g(\mathbf{x})$ is continuous at $\mathbf{x} = \mathbf{a}$ and
$f(u)$ is continuous at $u = g(\mathbf{a})$

then $\lim_{\mathbf{x}\to\mathbf{a}} f(g(\mathbf{x})) = f\left(\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x})\right) = f(g(\mathbf{a}))$

so $f \circ g(\mathbf{x}) = f(g(\mathbf{x}))$ is continuous at $\mathbf{a}$.

Example 10. Where is $f(x,y) = \ln\left(x^2 + y^2 - 3\right)$ continuous?

Solution. The function $\ln(u)$ is continuous for $u > 0$ and the function $x^2 + y^2 - 3$ is continuous on all of $\mathbb{R}^2$ so we need:

$$x^2 + y^2 - 3 > 0 \quad \Rightarrow \quad x^2 + y^2 > 3$$

hence $f(x,y) = \ln(x^2 + y^2 - 3)$ is continuous at all points *outside* a closed disk of radius $\sqrt{3}$ centered at $(0,0)$. ◀

Practice 7. Where is $g(x,y) = \arcsin\left(x^2 + y^2 - 3\right)$ continuous?

Using Polar Coordinates to Investigate Limits

For functions of two variables, converting from rectangular coordinates (x,y) to polar coordinates (r,θ) can sometimes provide greater insight into the behavior of a function near the origin. In Example 1, we could have written:

$$\frac{x^2y}{x^2+y^2} = \frac{(r\cos(\theta))^2\,(r\sin(\theta))}{(r\cos(\theta))^2+(r\sin(\theta))^2} = \frac{r^3\cos^2(\theta)\sin(\theta)}{r^2} = r\cos^2(\theta)\sin(\theta)$$

As $(x,y) \to (0,0)$, $|r| = \sqrt{x^2+y^2} \to 0$, so $\left|r\cos^2(\theta)\sin(\theta)\right| \le |r| \to 0$, showing that the limit, as $(x,y) \to (0,0)$, of the original expression must be 0 regardless of the path involved. Applying the same method to the expression in Example 2 yields:

$$\frac{x^2}{x^2+y^2} = \frac{(r\cos(\theta))^2}{(r\cos(\theta))^2+(r\sin(\theta))^2} = \frac{r^2\cos^2(\theta)}{r^2} = \cos^2(\theta)$$

so the value of this expression depends not on the distance of (x,y) from $(0,0)$ but rather on the path (x,y) follows as it approaches $(0,0)$.

Practice 8. Redo Practice 1 using polar coordinates.

Practice 9. Redo Practice 2 using polar coordinates.

Polar coordinates can only help us investigate limits when the limit point is $(0,0)$.

Example 11. Find $\displaystyle\lim_{(x,y)\to(5,-3)} \frac{(x-5)(y+3)^2}{(x-5)^2+(y+3)^2}$.

Solution. A change of variables with $u = x-5$ and $v = y+3$ yields:

$$\lim_{(x,y)\to(5,-3)} \frac{(x-5)(y+3)^2}{(x-5)^2+(y+3)^2} = \lim_{(u,v)\to(0,0)} \frac{u\cdot v^2}{u^2+v^2}$$

Now let $u = r\cos(\theta)$ and $v = \sin(\theta)$ so that the limit becomes:

$$\lim_{r\to 0^+} \frac{r\cos(\theta)\cdot r^2\sin^2(\theta)}{r^2} = \lim_{r\to 0^+} r\cos(\theta)\sin^2(\theta) = 0$$

no matter the value of θ. ◀

Limits and Continuity in Three (or More) Variables

Because of our use of the vector notation $\mathbf{x} \to \mathbf{a}$, the definitions and results about limits and continuity apply directly to functions with any number of variables.

Example 12. Compute $\lim_{(x,y,z)\to(7,3,\pi)} \cos(xyz)$.

Solution. The function $\cos(u)$ is everywhere continuous (on $\mathbb{R}$), as is the polynomial function xyz (on $\mathbb{R}^3$), so we can conclude that $\lim_{(x,y,z)\to(7,3,\pi)} \cos(xyz) = \cos(21\pi) = -1$. ◀

For limits of functions of three variables, converting to spherical coordinates often helps when the limit point is $(0,0,0)$.

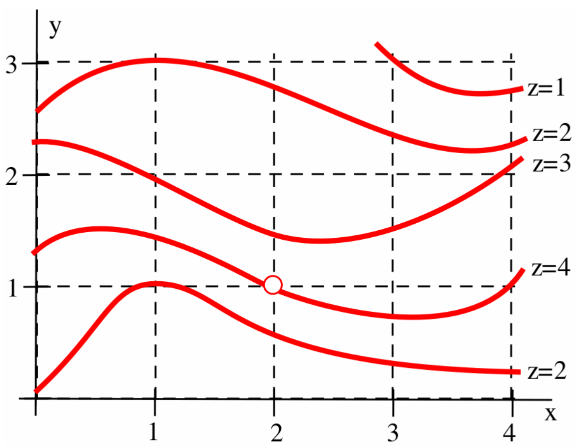
Practice 10. Where is $f(x,y,z) = \sqrt{x^2+y^2-z}$ continuous?

When evaluating limits of multivariable functions, keep in mind that showing that a limit does *not* exist merely requires finding two paths that yield different limits, while showing that a limit *does* exist requires an algebraic argument (possibly using polar or spherical coordinates).

14.2 Problems

In Problems 1–4, use the given level curves to determine each limit as best you can. (Assume each function behaves "nicely" away from the given level curves.)

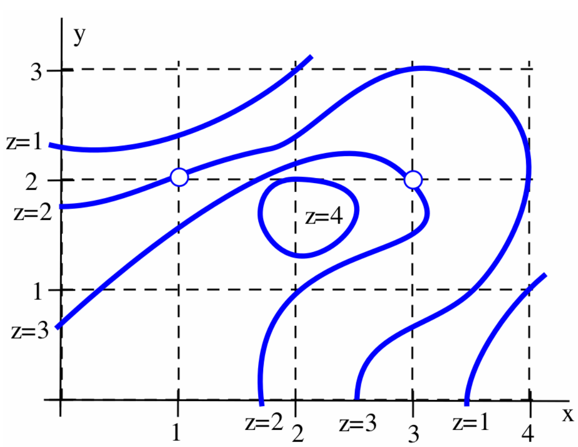
1. Several level curves of $z = f(x, y)$ appear below.



(a) $\lim_{(x,y)\to(1,2)} f(x, y)$ (b) $\lim_{(x,y)\to(1,1)} f(x, y)$

(c) $\lim_{(x,y)\to(2,1)} f(x, y)$ (d) $\lim_{(x,y)\to(3,2)} f(x, y)$

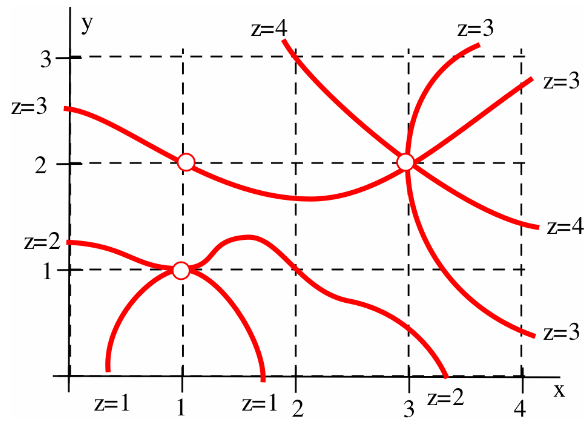
2. Several level curves of $z = g(x, y)$ appear below.



(a) $\lim_{(x,y)\to(2,2)} g(x, y)$ (b) $\lim_{(x,y)\to(2,1)} g(x, y)$

(c) $\lim_{(x,y)\to(1,2)} g(x, y)$ (d) $\lim_{(x,y)\to(3,2)} g(x, y)$

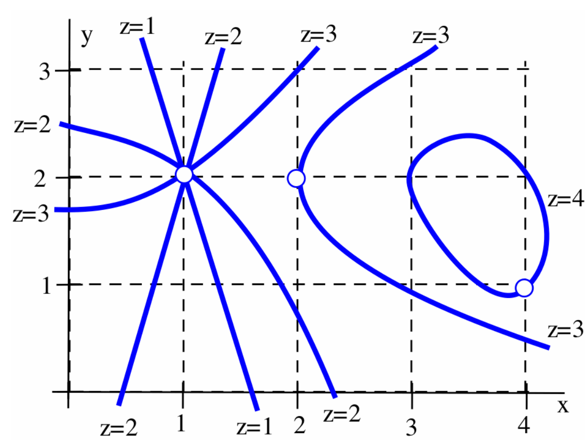
3. Several level curves of $z = S(x, y)$ appear below.



(a) $\lim_{(x,y)\to(1,2)} S(x, y)$ (b) $\lim_{(x,y)\to(2,1)} S(x, y)$

(c) $\lim_{(x,y)\to(1,1)} S(x, y)$ (d) $\lim_{(x,y)\to(3,2)} S(x, y)$

4. Several level curves of $z = T(x, y)$ appear below.



(a) $\lim_{(x,y)\to(3,3)} T(x, y)$ (b) $\lim_{(x,y)\to(2,2)} T(x, y)$

(c) $\lim_{(x,y)\to(1,2)} T(x, y)$ (d) $\lim_{(x,y)\to(4,1)} T(x, y)$

5. A friend tells you that:

$$\lim_{x \to 0} f(x,0) = \lim_{y \to 0} f(0,y) = 5$$

What can you conclude about $\lim_{(x,y)\to(0,0)} f(x,y)$?

6. A classmate tells you that:

$$\lim_{x \to 0} g(x,0) = 5 \quad \text{and} \quad \lim_{y \to 0} g(0,y) = -5$$

What can you conclude about $\lim_{(x,y)\to(0,0)} g(x,y)$?

In Problems 7–30, compute the limit if it exists or show that the limit does not exist.

7. $\lim_{(x,y)\to(2,3)} \left[x^2y^2 - 2xy^5 + 3y\right]$

8. $\lim_{(x,y)\to(-3,4)} \left[x^3 + 3x^2y^2 - 5y^3 + 1\right]$

9. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

10. $\lim_{(x,y)\to(-2,1)} \dfrac{x^2 + xy + y^2}{x^2 - y^2}$

11. $\lim_{(x,y)\to(\pi,\pi)} x \cdot \sin\left(\dfrac{x+y}{4}\right)$

12. $\lim_{(x,y)\to(1,4)} e^{\sqrt{x+2y}}$

13. $\lim_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}$

14. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x + y}$

15. $\lim_{(x,y)\to(0,0)} \dfrac{x - y}{x^2 + y^2}$

16. $\lim_{(x,y)\to(0,0)} \dfrac{x^3}{x^3 + y^3}$

17. $\lim_{(x,y)\to(0,0)} \dfrac{8x^2y^2}{x^4 + y^4}$

18. $\lim_{(x,y)\to(0,0)} \dfrac{x^3 + xy^2}{x^2 + y^2}$

19. $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2 + y^2}$

20. $\lim_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2 + y^2}$

21. $\lim_{(x,y)\to(0,0)} \dfrac{\sqrt{|xy|}}{\sqrt{x^2 + y^2}}$

22. $\lim_{(x,y)\to(0,0)} \dfrac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$

23. $\lim_{(x,y)\to(0,0)} \dfrac{xy + 1}{x^2 + y^2 + 1}$

24. $\lim_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2 + y^6}$

25. $\lim_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^4 + y^2}$

26. $\lim_{(x,y)\to(0,0)} \dfrac{x^3y^2}{x^2 + y^2}$

27. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$

28. $\lim_{(x,y)\to(0,0)} \dfrac{\sqrt{x^2 + y^2 + 1} - 1}{x^2 + y^2}$

29. $\lim_{(x,y)\to(0,1)} \dfrac{xy - x}{x^2 + y^2 - 2x + 2y + 2}$

30. $\lim_{(x,y)\to(1,-1)} \dfrac{x^2 + y^2 - 2x - 2y}{x^2 + y^2 - 2x + 2y + 2}$

31. Use the paths $\mathbf{r}_1(t) = \langle t, 0, 0\rangle$ and $\mathbf{r}_2(t) = \langle t, t, t\rangle$ to investigate:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2 + y^2 + z^2}$$

What can you conclude about this limit?

32. Use the paths $\mathbf{r}_1(t) = \langle t, 0, 0\rangle$ and $\mathbf{r}_2(t) = \langle t, t, t\rangle$ to investigate:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2}{x^2 + y^2 + z^2}$$

What can you conclude about this limit?

In Problems 33–38, compute the limit if it exists or show that the limit does not exist.

33. $\lim_{(x,y,z)\to(1,2,3)} \dfrac{xz^2 - y^2z}{xyz - 1}$

34. $\lim_{(x,y,z)\to(2,3,0)} \left[x \cdot e^x + \ln(2x - y)\right]$

35. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2 - y^2 - z^2}{x^2 + y^2 + z^2}$

36. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz + xz}{x^2 + y^2 + z^2}$

37. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz^2 + xz^2}{x^2 + y^2 + z^2}$

38. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2y^2z^2}{x^2 + y^2 + z^2}$

39. The function f from Problem 1 is undefined at $(2,1)$. Specify a value for $f(2,1)$ so that f will be continuous at $(2,1)$.

40. The function g from Problem 2 is undefined at $(1,2)$ and $(3,2)$. Can you specify values for $g(1,2)$ and $g(3,2)$ so that g will be continuous at each of those points?

41. The function S from Problem 3 is undefined at $(1,1)$, $(1,2)$ and $(3,2)$. Can you specify values for $S(1,1)$, $S(1,2)$ and $S(3,2)$ so that S will be continuous at each of those points?

42. The function T from Problem 4 is undefined at $(1,2)$, $(2,2)$ and $(4,1)$. Can you specify values for $T(1,2)$, $T(2,2)$ and $T(4,1)$ so T is continuous at each of those points?

In Problems 43–54, determine where the given function is continuous.

43. $f(x,y) = \ln(2x+3y)$

44. $g(x,y) = e^{xy}\sin(x+y)$

45. $F(x,y) = \dfrac{x^2+y^2+1}{x^2+y^2-1}$

46. $G(x,y) = \dfrac{x^6+x^3y^3+y^6}{x^3+y^3}$

47. $S(x,y) = \sqrt{x+y} - \sqrt{x-y}$

48. $T(x,y) = 2^{x\tan(y)}$

49. $\varphi(x,y) = \lfloor xy \rfloor$

50. $\mu(x,y) = \lfloor x \rfloor + \lfloor y \rfloor$

51. $\Phi(x,y) = \lfloor x+y \rfloor$

52. $\Theta(x,y) = \lfloor \cos(x+y) \rfloor$

53. $f(x,y,z) = x\ln(yz)$

54. $g(x,y,z) = x + y\sqrt{x+z}$

Open and Closed Sets

The definitions for limits and continuity of a function of one variable involve open intervals of the form $|x-a| < \delta$, which we can express as $a-\delta < x < a+\delta$. These intervals are called open because they contain neither endpoint. An interval of the form $a-\delta \le x \le a+\delta$ is closed and one of the form $a-\delta < x \le a+\delta$ is neither open nor closed.

For functions of two or more variables, the definitions in this section involve sets of the form $\|\mathbf{x}-\mathbf{a}\| < \delta$. In two dimensions, we can rewrite this inequality in the form $\|\langle x,y\rangle - \langle x_0,y_0\rangle\| < \delta$ or:

$$\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \quad \Rightarrow \quad (x-x_0)^2+(y-y_0)^2 < \delta^2$$

which describes an open disk with center (x_0,y_0), sitting "inside" the circle $(x-x_0)^2+(y-y_0)^2 = \delta^2$. The disk is open because it does not contain its boundary (the circle), just as an interval is open when it does not contain its boundary (the left and right endpoints of the interval).

Similarly, in three dimensions our definitions in this section involved an open **ball** of the form $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 < \delta^2$ sitting inside the sphere $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = \delta^2$. Many regions of interest are not intervals, disks or balls, however, so we will need a definition of "open" and "closed" for more complicated sets.

In $\mathbb{R}^n$, a point $\mathbf{x}$ is in the **interior** of a region $\mathcal{R}$ if there is a (possibly very small) number $\delta > 0$ so that a ball of radius δ centered at $\mathbf{x}$ contains only points in $\mathcal{R}$. A point $\mathbf{x}$ is on the **boundary** of $\mathcal{R}$ if any ball centered at $\mathbf{x}$ (no matter how small the radius) contains both points inside $\mathcal{R}$ and points outside $\mathcal{R}$. A set $\mathcal{R}$ is **open** if it contains none of its boundary points and **closed** if it contains all of its boundary points; otherwise the set is "neither open nor closed."

14.2 Practice Answers

1. If (x, y) approaches $(0,0)$ along the x-axis, where $y = 0$:
$$F(x,0) = \frac{x^2 - 0^2}{x^2 + 0^2} = \frac{x^2}{x^2} = 1$$
for $x \neq 0$. Along the y-axis, where $x = 0$:
$$F(0,y) = \frac{0^2 - y^2}{0^2 + y^2} = \frac{-y^2}{y^2} = -1$$
for $y \neq 0$, so $F(x, y)$ has no single limiting value as $(x, y) \to (0,0)$.

2. Substituting mx for y:
$$G(x,mx) = \frac{x \cdot mx}{x^2 + (mx)^2} = \frac{mx^2}{(1+m)x^2} = \frac{m}{1+m^2}$$
when $x \neq 0$. The limiting value of $G(x, y)$ depends on the line to which we restrict the values of (x, y).

3. Substituting Ax^2 for y:
$$f\left(x, Ax^2\right) = \frac{x^2 \cdot Ax^2}{x^4 + (Ax^2)^2} = \frac{Ax^4}{(1+A^2)x^4} = \frac{A^2}{1+A^2}$$
when $x \neq 0$. The limiting value of $f(x, y)$ depends on the parabola to which we restrict the values of (x, y).

4. Applying the Main Limit Theorem and the results of Example 6:
$$\lim_{(x,y)\to(2,-5)} \frac{3x^2 + 6xy - 7y^2 + 12}{(x+2)^2 + (y-5)^2} = \frac{3 \cdot 2^2 + 6 \cdot 2(-5) - 7(-5)^2 + 12}{(2+2)^2 + (-5-5)^2}$$
which simplifies to $-\dfrac{211}{116}$.

5. Because $g(x, y)$ is a rational function, it is continuous for all (x, y) except where the denominator is 0, and that occurs only at $(-2, 5)$.

6. Because $G(x, y)$ is a rational function, it is continuous everywhere except at the points where its denominator equals 0, and this occurs only at $(0,0)$. From Example 1 we know that $\lim_{(x,y)\to(0,0)} G(x, y) = 0$, but $G(0,0) = 1 \neq 0$, so $G(x, y)$ is not continuous at $(0,0)$.

7. Because $x^2 + y^2 - 3$ is a polynomial, it is continuous everywhere. The domain of $\arcsin(u)$ is $[-1, 1]$ and $\arcsin(u)$ is continuous on $(-1, 1)$, hence we need:
$$-1 < x^2 + y^2 - 3 < 1 \;\Rightarrow\; 2 < x^2 + y^2 < 4$$
so $g(x, y)$ is continuous on the annulus consisting of all points inside a circle of radius 2 centered at $(0,0)$ and outside a circle of radius $\sqrt{2}$ centered at $(0,0)$.

8. Substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$:

$$F(x,y) = F\left(r\cos(\theta), r\sin(\theta)\right) = \frac{r^2\cos^2(\theta) - r^2\sin^2(\theta)}{r^2\cos^2(\theta) + r^2\sin^2(\theta)} = \cos(2\theta)$$

which shows that the limiting values of $F(x,y)$ as $(x,y) \to (0,0)$ depend not on the distance of (x,y) from $(0,0)$ but on the path along which (x,y) moves toward $(0,0)$.

9. Substituting $x = r\cos(\theta)$ and $y = r\sin(\theta)$:

$$G(x,y) = F\left(r\cos(\theta), r\sin(\theta)\right) = \frac{r\cos(\theta)\cdot r\sin(\theta)}{r^2\cos^2(\theta) + r^2\sin^2(\theta)} = \frac{1}{2}\sin(2\theta)$$

which shows that the limiting values of $G(x,y)$ as $(x,y) \to (0,0)$ depend not on the distance of (x,y) from $(0,0)$ but on the path along which (x,y) moves toward $(0,0)$.

10. Because $x^2 + y^2 - z$ is a polynomial, it is continuous everywhere. The domain of $\sqrt{u}$ is $[0,\infty)$ and $\sqrt{u}$ is continuous on $(0,\infty)$, hence we need $0 < x^2 + y^2 - z \Rightarrow z < x^2 + y^2$, so $f(x,y,z)$ is continuous on the region in $\mathbb{R}^3$ below the paraboloid $z = x^2 + y^2$.