2

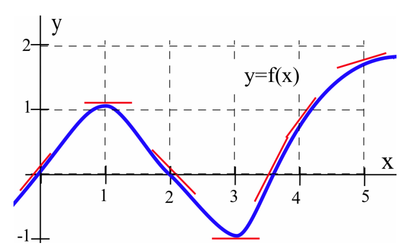
The Derivative

The two previous chapters have laid the foundation for the study of calculus. They provided a review of some material you will need and started to emphasize the various ways we will view and use functions: functions given by graphs, equations and tables of values.

Chapter 2 will focus on the idea of tangent lines. We will develop a definition for the derivative of a function and calculate derivatives of some functions using this definition. Then we will examine some of the properties of derivatives, see some relatively easy ways to calculate the derivatives, and begin to look at some ways we can use them.

2.0 *Introduction to Derivatives*

This section begins with a very graphical approach to slopes of tangent lines. It then examines the problem of finding the slopes of the tangent lines for a single function, $y = x^2$, in some detail — and illustrates how these slopes can help us solve fairly sophisticated problems.

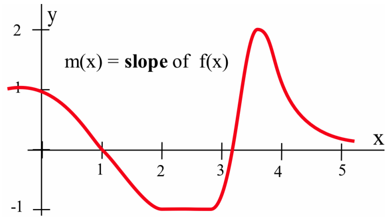


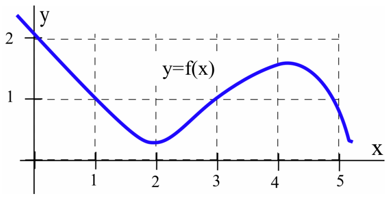
Slopes of Tangent Lines: Graphically

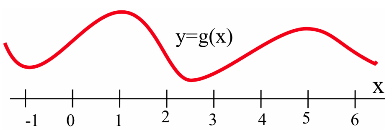
The figure in the margin shows the graph of a function $y = f(x)$. We can use the information in the graph to fill in the table:

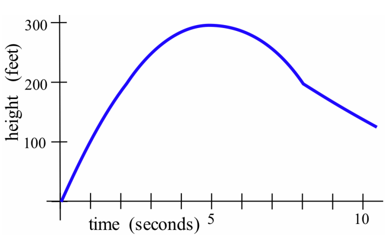
x	$y = f(x)$	$m(x)$
0	0	1
1	1	0
2	0	-1
3	-1	0
4	1	1
5	2	$\frac{1}{2}$

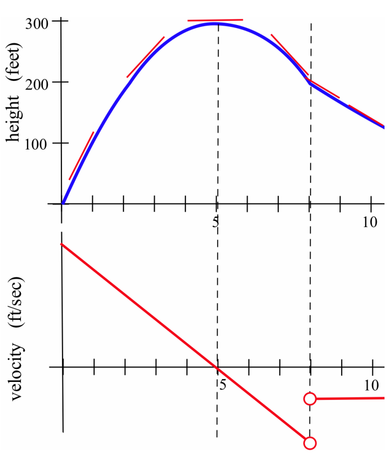
where $m(x)$ is the (estimated) **slope** of the line tangent to the graph of $y = f(x)$ at the point (x, y). We can estimate the values of $m(x)$ at some non-integer values of x as well: $m(0.5) \approx 0.5$ and $m(1.3) \approx -0.3$,











for example. We can even say something about the behavior of $m(x)$ over entire intervals: if $0 < x < 1$, then $m(x)$ is positive, for example.

The values of $m(x)$ definitely depend on the values of x (the slope varies as x varies, and there is at most one slope associated with each value of x) so $m(x)$ is a function of x. We can use the results in the table to help sketch a graph of the function $m(x)$ (see top margin figure).

Practice 1. A graph of $y = f(x)$ appears in the margin. Set up a table of (estimated) values for x and $m(x)$, the slope of the line tangent to the graph of $y = f(x)$ at the point (x, y), and then sketch a graph of the function $m(x)$.

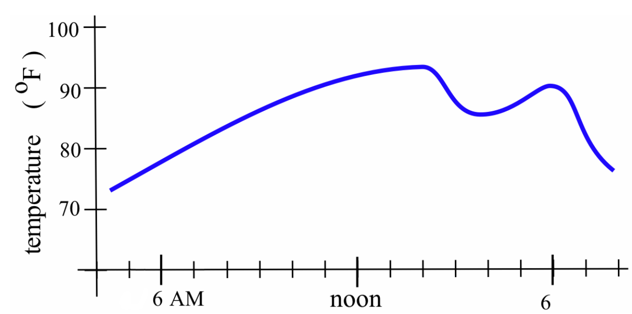
In some applications, we need to know where the graph of a function $f(x)$ has horizontal tangent lines (that is, where the slope of the tangent line equals 0). The slopes of the lines tangent to graph of $y = f(x)$ in Practice 1 are 0 when $x = 2$ or $x \approx 4.25$.

Practice 2. At what values of x does the graph of $y = g(x)$ (in the margin) have horizontal tangent lines?

Example 1. The graph of the height of a rocket at time t appears in the margin. Sketch a graph of the **velocity** of the rocket at time t. (Remember that instantaneous velocity corresponds to the **slope of the line tangent** to the graph of position or height function.)

Solution. The penultimate margin figure shows some sample tangent line segments, while the bottom margin figure shows the velocity of the rocket. (What so you think happened at time $t = 8$?) ◀

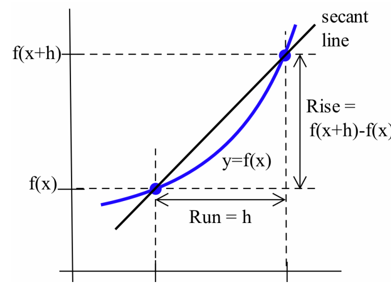
Practice 3. The graph below shows the temperature during a summer day in Chicago. Sketch a graph of the **rate** at which the temperature is changing at each moment in time. (As with instantaneous velocity, the instantaneous rate of change for the temperature corresponds to the slope of the line tangent to the temperature graph.)



The function $m(x)$, the slope of the line tangent to the graph of $y = f(x)$ at $(x, f(x))$, is called the **derivative** of $f(x)$.

We used the idea of the slope of the tangent line all throughout Chapter 1. In Section 2.1, we will formally define the derivative of a function and begin to examine some of its properties, but first let's see what we can do when we have a formula for $f(x)$.

Tangents to $y = x^2$



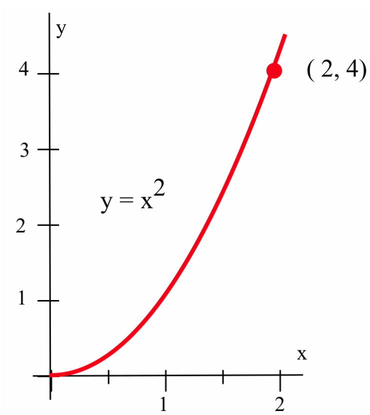
When we have a formula for a function, we can determine the slope of the tangent line at a point $(x, f(x))$ by calculating the slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$:

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

and then taking the limit of m_{sec} as h approaches 0:

$$m_{\text{tan}} = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

Example 2. Find the slope of the line tangent to the graph of the function $y = f(x) = x^2$ at the point $(2, 4)$.



Solution. In this example, $x = 2$, so $x + h = 2 + h$ and $f(x+h) = f(2+h) = (2+h)^2$. The slope of the tangent line at $(2, 4)$ is

$$\begin{aligned} m_{\text{tan}} &= \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{(2+h) - (2)} \\ &= \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} \\ &= \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} [4 + h] = 4 \end{aligned}$$

The line tangent to $y = x^2$ at the point $(2, 4)$ has slope 4. ◀

We can use the point-slope formula for a line to find an equation of this tangent line:

$$y - y_0 = m(x - x_0) \Rightarrow y - 4 = 4(x - 2) \Rightarrow y = 4x - 4$$

Practice 4. Use the method of Example 2 to show that the slope of the line tangent to the graph of $y = f(x) = x^2$ at the point $(1, 1)$ is $m_{\text{tan}} = 2$. Also find the values of m_{tan} at $(0, 0)$ and $(-1, 1)$.

It is possible to compute the slopes of the tangent lines one point at a time, as we have been doing, but that is not very efficient. You should have noticed in Practice 4 that the algebra for each point was very similar, so let's do all the work just once, for an arbitrary point $(x, f(x)) = (x, x^2)$ and then use the general result to find the slopes at the particular points we're interested in.

The slope of the line tangent to the graph of $y = f(x) = x^2$ at the arbitrary point (x, x^2) is:

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} m_{\sec} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{(x+h)-(x)} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} [2x+h] = 2x \end{aligned}$$

The slope of the line tangent to the graph of $y = f(x) = x^2$ at the point (x, x^2) is $m_{\tan} = 2x$. We can use this general result at any value of x without going through all of the calculations again. The slope of the line tangent to $y = f(x) = x^2$ at the point $(4, 16)$ is $m_{\tan} = 2(4) = 8$ and the slope at (p, p^2) is $m_{\tan} = 2(p) = 2p$. The value of x determines the location of our point on the curve, (x, x^2), as well as the slope of the line tangent to the curve at that point, $m_{\tan} = 2x$. The slope $m_{\tan} = 2x$ is a **function** of x and is called the **derivative** of $y = x^2$.

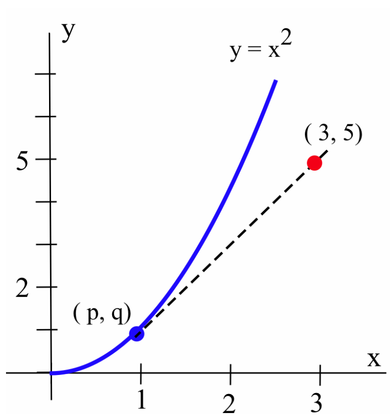
Simply knowing that the slope of the line tangent to the graph of $y = x^2$ is $m_{\tan} = 2x$ at a point (x, y) can help us quickly find an equation of the line tangent to the graph of $y = x^2$ at any point and answer a number of difficult-sounding questions.

Example 3. Find equations of the lines tangent to $y = x^2$ at the points $(3, 9)$ and (p, p^2).

Solution. At $(3, 9)$, the slope of the tangent line is $2x = 2(3) = 6$, and the equation of the line is $y - 9 = 6(x - 3) \Rightarrow y = 6x - 9$.

At (p, p^2), the slope of the tangent line is $2x = 2(p) = 2p$, and the equation of the line is $y - p^2 = 2p(x - p) \Rightarrow y = 2px - p^2$. ◀

Example 4. A rocket has been programmed to follow the path $y = x^2$ in space (from left to right along the curve, as seen in the margin figure), but an emergency has arisen and the crew must return to their base, which is located at coordinates $(3, 5)$. At what point on the path $y = x^2$ should the captain turn off the engines so that the ship will coast along a path tangent to the curve to return to the base?



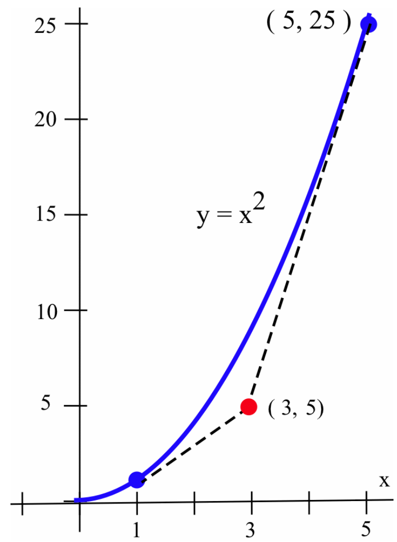
Solution. You might spend a few minutes trying to solve this problem without using the relation $m_{\tan} = 2x$, but the problem is much easier if we do use that result.

Let's assume that the captain turns off the engine at the point (p, q) on the curve $y = x^2$ and then try to determine what values p and q must have so that the resulting tangent line to the curve will go through the point $(3, 5)$. The point (p, q) is on the curve $y = x^2$, so $q = p^2$ and the equation of the tangent line, found in Example 3, must then be $y = 2px - p^2$.

To find the value of p so that the tangent line will go through the point $(3,5)$, we can substitute the values $x = 3$ and $y = 5$ into the equation of the tangent line and solve for p:

$$y = 2px - p^2 \Rightarrow 5 = 2p(3) - p^2 \Rightarrow p^2 - 6p + 5 = 0$$
$$\Rightarrow (p-1)(p-5) = 0$$

The only solutions are $p = 1$ and $p = 5$, so the only possible points are $(1,1)$ and $(5,25)$. You can verify that the tangent lines to $y = x^2$ at $(1,1)$ and $(5,25)$ both go through the point $(3,5)$. Because the ship is moving from left to right along the curve, the captain should turn off the engines at the point $(1,1)$. (Why not at $(5,25)$?) ◀

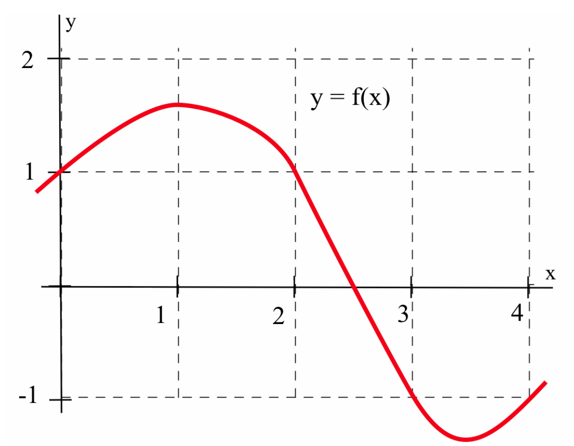


Practice 5. Verify that if the rocket engines in Example 4 are shut off at $(2,4)$, then the rocket will go through the point $(3,8)$.

2.0 Problems

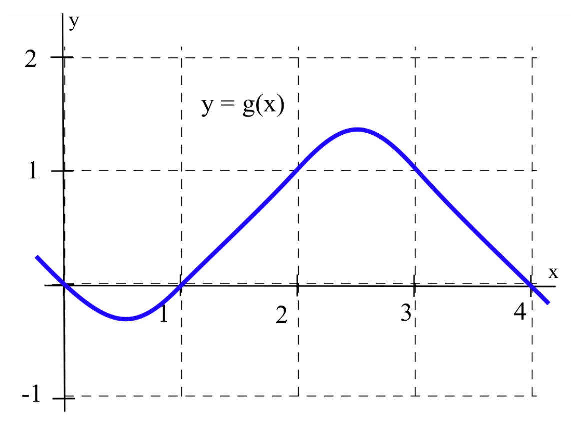
1. Use the function $f(x)$ graphed below to fill in the table and then graph $m(x)$, the estimated slope of the tangent line to $y = f(x)$ at the point (x,y).

x	$f(x)$	$m(x)$	x	$f(x)$	$m(x)$
0.0			2.5		
0.5			3.0		
1.0			3.5		
1.5			4.0		
2.0					



2. Use the function $g(x)$ graphed below to fill in the table and then graph $m(x)$, the estimated slope of the tangent line to $y = g(x)$ at the point (x,y).

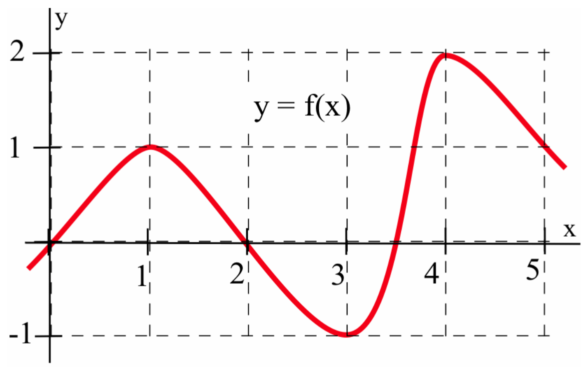
x	$g(x)$	$m(x)$	x	$g(x)$	$m(x)$
0.0			2.5		
0.5			3.0		
1.0			3.5		
1.5			4.0		
2.0					



3. (a) At what values of x does the graph of f (shown below) have a horizontal tangent line?

   (b) At what value(s) of x is the value of f the largest? Smallest?

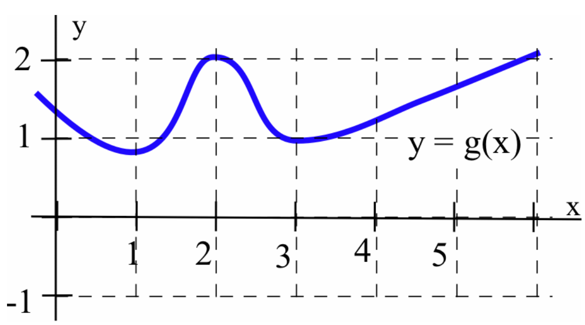
   (c) Sketch a graph of $m(x)$, the slope of the line tangent to the graph of f at the point $(x, f(x))$.



4. (a) At what values of x does the graph of g (shown below) have a horizontal tangent line?

   (b) At what value(s) of x is the value of g the largest? Smallest?

   (c) Sketch a graph of $m(x)$, the slope of the line tangent to the graph of g at the point $(x, g(x))$.



5. (a) Sketch the graph of $f(x) = \sin(x)$ on the interval $-3 \le x \le 10$.

   (b) Sketch a graph of $m(x)$, the slope of the line tangent to the graph of $\sin(x)$ at the point $(x, \sin(x))$.

   (c) Your graph in part (b) should look familiar. What function is it?

6. Match the situation descriptions with the corresponding time-velocity graphs shown below.

   (a) A car quickly leaving from a stop sign.

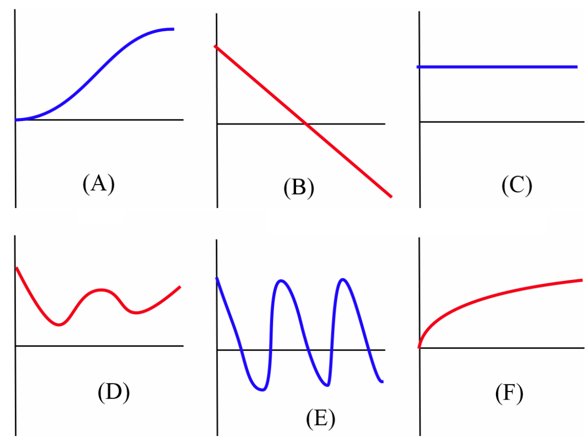
   (b) A car sedately leaving from a stop sign.

   (c) A student bouncing on a trampoline.

   (d) A ball thrown straight up.

   (e) A student confidently striding across campus to take a calculus test.

   (f) An unprepared student walking across campus to take a calculus test.



Problems 7–10 assume that a rocket is following the path $y = x^2$, from left to right.

7. At what point should the engine be turned off in order to coast along the tangent line to a base at $(5, 16)$?

8. At $(3, -7)$?

9. At $(1, 3)$?

10. Which points in the plane can not be reached by the rocket? Why not?

In Problems 11–16, perform these steps:

(a) Calculate and simplify:

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

(b) Determine $m_{\tan} = \lim_{h \to 0} m_{\text{sec}}$.

(c) Evaluate $m_{\tan}$ at $x = 2$.

(d) Find an equation of the line tangent to the graph of f at $(2, f(2))$.

11. $f(x) = 3x - 7$

12. $f(x) = 2 - 7x$

13. $f(x) = ax + b$ where a and b are constants

14. $f(x) = x^2 + 3x$

15. $f(x) = 8 - 3x^2$

16. $f(x) = ax^2 + bx + c$ where a, b and c are constants

In Problems 17–18, use the result:

$$f(x) = ax^2 + bx + c \Rightarrow m_{\text{tan}} = 2ax + b$$

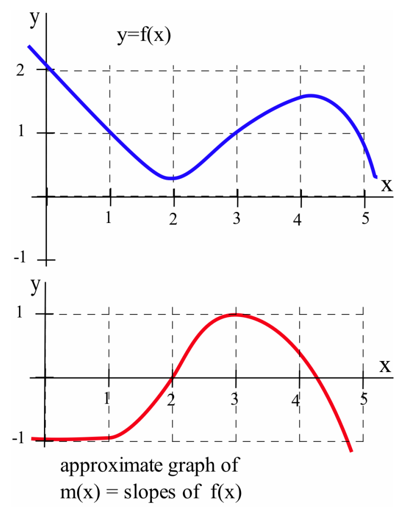
17. Given $f(x) = x^2 + 2x$, at which point(s) $(p, f(p))$ does the line tangent to the graph at that point also go through the point $(3,6)$?

18. (a) If $a \neq 0$, then what is the shape of the graph of $y = f(x) = ax^2 + bx + c$?
    (b) At what value(s) of x is the line tangent to the graph of $f(x)$ horizontal?

2.0 Practice Answers

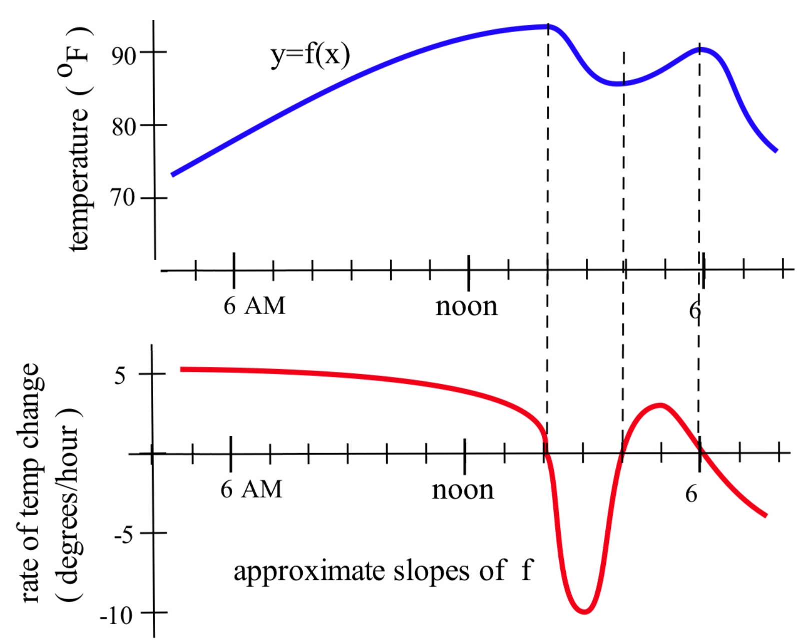
1. Approximate values of $m(x)$ appear in the table in the margin; the margin figure shows a graph of $m(x)$.

x	$f(x)$	$m(x)$
0	2	-1
1	1	-1
2	$\frac{1}{3}$	0
3	1	1
4	$\frac{3}{2}$	$\frac{1}{2}$
5	1	-2



2. The tangent lines to the graph of g are horizontal (slope $= 0$) when $x \approx -1$, 1, 2.5 and 5.

3. The figure below shows a graph of the approximate rate of temperature change (slope).



4. At $(1,1)$, the slope of the tangent line is:

$$\begin{aligned} m_{\text{tan}} &= \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - (1)} \\ &= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} = \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} [2 + h] = 2 \end{aligned}$$

so the line tangent to $y = x^2$ at the point $(1,1)$ has slope 2. At $(0,0)$:

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} m_{\sec} = \lim_{h\to 0} \frac{f(0+h) - f(1)}{(0+h) - (0)} \\ &= \lim_{h\to 0} \frac{(0+h)^2 - 0^2}{h} = \lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0 \end{aligned}$$

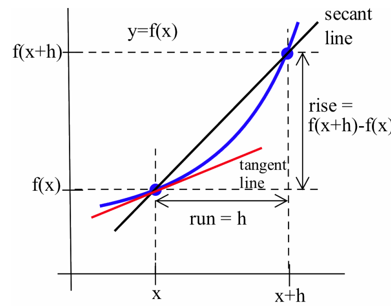
so the line tangent to $y = x^2$ at $(0,0)$ has slope 0. At $(-1,1)$:

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0} m_{\sec} = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{(-1+h) - (-1)} \\ &= \lim_{h\to 0} \frac{(-1+h)^2 - (-1)^2}{h} = \lim_{h\to 0} \frac{1 - 2h + h^2 - 1}{h} \\ &= \lim_{h\to 0} \frac{-2h + h^2}{h} = \lim_{h\to 0} [-2 + h] = -2 \end{aligned}$$

so the line tangent to $y = x^2$ at the point $(-1,1)$ has slope -2.

5. From Example 4 we know the slope of the tangent line is $m_{\tan} = 2x$, so the slope of the tangent line at $(2,4)$ is $m_{\tan} = 2x = 2(2) = 4$. The tangent line has slope 4 and goes through the point $(2,4)$, so an equation of the tangent line (using $y - y_0 = m(x - x_0)$) is $y - 4 = 4(x - 2)$ or $y = 4x - 4$. The point $(3,8)$ satisfies the equation $y = 4x - 4$, so the point $(3,8)$ lies on the tangent line.

2.1 *The Definition of Derivative*



The graphical idea of a **slope of a tangent line** is very useful, but for some purposes we need a more algebraic definition of the **derivative of a function**. We will use this definition to calculate the derivatives of several functions and see that these results agree with our graphical understanding. We will also look at several different interpretations for the derivative, and obtain a theorem that will allow us to easily and quickly determine the derivative of any fixed power of x.

In the previous section we found the slope of the tangent line to the graph of the function $f(x) = x^2$ at an arbitrary point $(x, f(x))$ by calculating the slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$:

$$m_{\text{sec}} = \frac{f(x+h) - f(x)}{(x+h) - (x)}$$

and then taking the limit of m_{sec} as h approached 0 (see margin). That approach to calculating slopes of tangent lines motivates the definition of the derivative of a function.

Definition of the Derivative:
The derivative of a function f is a new function, f' (pronounced "eff prime"), whose value at x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if this limit exists and is finite.

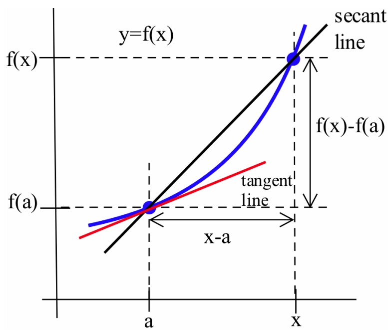
This is **the** definition of differential calculus, and you must know it and understand what it says. The rest of this chapter and all of Chapter 3 are built on this definition, as is much of what appears in later chapters. It is remarkable that such a simple idea (the slope of a tangent line) and such a simple definition (for the derivative f') will lead to so many important ideas and applications.

Notation

There are three commonly used notations for the derivative of $y = f(x)$:

- $f'(x)$ emphasizes that the derivative is a function related to f
- $\mathbf{D}(f)$ emphasizes that we perform an operation on f to get f'
- $\dfrac{df}{dx}$ emphasizes that the derivative is the limit of $\dfrac{\Delta f}{\Delta x} = \dfrac{f(x+h) - f(x)}{h}$

We will use all three notations so that you can become accustomed to working with each of them.



The function $f'(x)$ gives the slope of the tangent line to the graph of $y = f(x)$ at the point $(x, f(x))$, or the instantaneous rate of change of the function f at the point $(x, f(x))$.

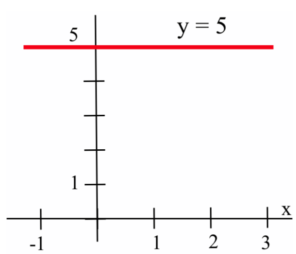
If, in the margin figure, we let x be the point $a + h$, then $h = x - a$. As $h \to 0$, we see that $x \to a$ and:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We will use whichever of these two forms is more convenient algebraically in a particular situation.

Calculating Some Derivatives Using the Definition

Fortunately, we will soon have some quick and easy ways to calculate most derivatives, but first we will need to use the definition to determine the derivatives of a few basic functions. In Section 2.2, we will use those results and some properties of derivatives to calculate derivatives of combinations of the basic functions. Let's begin by using the graphs and then the definition to find a few derivatives.



Example 1. Graph $y = f(x) = 5$ and estimate the **slope** of the tangent line at each point on the graph. Then use the definition of the derivative to calculate the exact slope of the tangent line at each point. Your graphical estimate and the exact result from the definition should agree.

Solution. The graph of $y = f(x) = 5$ is a horizontal line (see margin), which has slope 0, so we should expect that its tangent line will also have slope 0.

Using the definition: With $f(x) = 5$, then $f(x + h) = 5$ no matter what h is, so:

$$\mathbf{D}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{5-5}{h} = \lim_{h \to 0} 0 = 0$$

and this agrees with our graphical estimate of the derivative. ◀

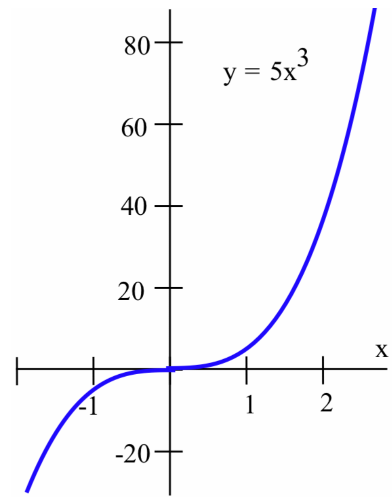
Using similar steps, it is easy to show that the derivative of *any* constant function is 0.

Theorem: If $f(x) = k$, then $f'(x) = 0$.

Practice 1. Graph $y = f(x) = 7x$ and estimate the slope of the tangent line at each point on the graph. Then use the definition of the derivative to calculate the exact slope of the tangent line at each point.

Example 2. Describe the derivative of $y = f(x) = 5x^3$ graphically and compute it using the definition. Find an equation of the line tangent to $y = 5x^3$ at the point $(1, 5)$.

Solution. It appears from the graph of $y = f(x) = 5x^3$ (see margin) that $f(x)$ is increasing, so the slopes of the tangent lines are positive except perhaps at $x = 0$, where the graph seems to flatten out.



With $f(x) = 5x^3$ we have:

$$f(x+h) = 5(x+h)^3 = 5(x^3 + 3x^2h + 3xh^2 + h^3)$$

and using this last expression in the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{5(x^3 + 3x^2h + 3xh^2 + h^3) - 5x^3}{h} \\ &= \lim_{h\to 0} \frac{15x^2h + 15xh^2 + 5h^3}{h} = \lim_{h\to 0} (15x^2 + 15xh + 5h^2) = 15x^2 \end{aligned}$$

so $\mathbf{D}(5x^3) = 15x^2$, which is positive except when $x = 0$ (as we predicted from the graph).

The function $f'(x) = 15x^2$ gives the slope of the line tangent to the graph of $f(x) = 5x^3$ at the point $(x, f(x))$. At the point $(1, 5)$, the slope of the tangent line is $f'(1) = 15(1)^2 = 15$. From the point-slope formula, an equation of the tangent line to f at that point is $y - 5 = 15(x - 1)$ or $y = 15x - 10$. ◀

Practice 2. Use the definition to show that the derivative of $y = x^3$ is $\frac{dy}{dx} = 3x^2$. Find an equation of the line tangent to the graph of $y = x^3$ at the point $(2, 8)$.

If f has a derivative at x, we say that f is **differentiable** at x. If we have a point on the graph of a differentiable function and a slope (the derivative evaluated at the point), it is easy to write an equation of the tangent line.

Tangent Line Formula:

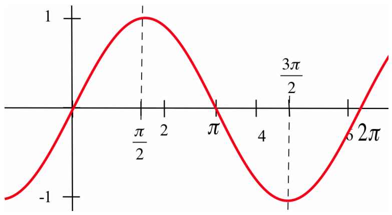
If $f(x)$ is differentiable at $x = a$
then an equation of the line tangent to f at $(a, f(a))$ is:

$$y = f(a) + f'(a)(x - a)$$

Proof. The tangent line goes through the point $(a, f(a))$ with slope $f'(a)$ so, using the point-slope formula, $y - f(a) = f'(a)(x - a)$ or $y = f(a) + f'(a)(x - a)$. □

Practice 3. The derivatives $\mathbf{D}(x) = 1$, $\mathbf{D}(x^2) = 2x$, $\mathbf{D}(x^3) = 3x^2$ exhibit the start of a pattern. Without using the definition of the derivative, what do you think the following derivatives will be? $\mathbf{D}(x^4)$, $\mathbf{D}(x^5)$, $\mathbf{D}(x^{43})$, $\mathbf{D}(\sqrt{x}) = \mathbf{D}(x^{\frac{1}{2}})$ and $\mathbf{D}(x^\pi)$. (Just make an intelligent "guess" based on the pattern of the previous examples.)

Before further investigating the "pattern" for the derivatives of powers of x and general properties of derivatives, let's compute the derivatives of two functions that are not powers of x: $\sin(x)$ and $|x|$.



> Theorem: $\mathbf{D}(\sin(x)) = \cos(x)$

The graph of $y = f(x) = \sin(x)$ (see margin) should be very familiar to you. The graph has horizontal tangent lines (slope = 0) when $x = \pm\frac{\pi}{2}$ and $x = \pm\frac{3\pi}{2}$ and so on. If $0 < x < \frac{\pi}{2}$, then the slopes of the tangent lines to the graph of $y = \sin(x)$ are positive. Similarly, if $\frac{\pi}{2} < x < \frac{3\pi}{2}$, then the slopes of the tangent lines are negative. Finally, because the graph of $y = \sin(x)$ is periodic, we expect that the derivative of $y = \sin(x)$ will also be periodic. Note that the function $\cos(x)$ possesses all of those desired properties for the slope function.

Proof. With $f(x) = \sin(x)$, apply an angle addition formula to get:

$$f(x+h) = \sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$$

and use this formula in the definition of the derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(\sin(x)\cos(h) + \cos(x)\sin(h)) - \sin(x)}{h} \end{aligned}$$

This limit looks formidable, but just collect the terms containing $\sin(x)$:

$$\lim_{h \to 0} \frac{(\sin(x)\cos(h) - \sin(x)) + \cos(x)\sin(h)}{h}$$

so you can factor out $\sin(x)$ from the first two terms, rewriting as:

$$\lim_{h \to 0} \left[\sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h}\right]$$

Now calculate the limits separately:

$$\lim_{h \to 0} \sin(x) \cdot \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \lim_{h \to 0} \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h}$$

The first and third limits do not depend on h, and we calculated the second and fourth limits in Section 1.2:

$$\sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x)$$

So $\mathbf{D}(\sin(x)) = \cos(x)$ and the various properties we expected of the derivative of $y = \sin(x)$ by examining its graph are true of $\cos(x)$. $\square$

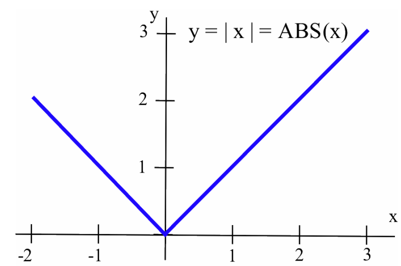
You will need the angle addition formula for cosine to rewrite $\cos(x+h)$ as:

$$\cos(x) \cdot \cos(h) - \sin(x) \cdot \sin(h)$$

Practice 4. Show that $\mathbf{D}(\cos(x)) = -\sin(x)$ using the definition.

The derivative of $\cos(x)$ resembles the situation for $\sin(x)$ but differs by an important negative sign. You should memorize both of these important derivatives.

Example 3. For $y = |x|$, find $\dfrac{dy}{dx}$.



Solution. The graph of $y = f(x) = |x|$ (see margin) is a "V" shape with its vertex at the origin. When $x > 0$, the graph is just $y = |x| = x$, which is part of a line with slope $+1$, so we should expect the derivative of $|x|$ to be $+1$. When $x < 0$, the graph is $y = |x| = -x$, which is part of a line with slope -1, so we expect the derivative of $|x|$ to be -1. When $x = 0$, the graph has a corner, and we should expect the derivative of $|x|$ to be undefined at $x = 0$, as there is no single candidate for a line tangent to the graph there.

Using the definition, consider the same three cases discussed previously: $x > 0$, $x < 0$ and $x = 0$.

If $x > 0$, then, for small values of h, $x + h > 0$, so:

$$\mathbf{D}(f(x)) = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

If $x < 0$, then, for small values of h, $x + h < 0$, so:

$$\mathbf{D}(f(x)) = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \lim_{h\to 0} \frac{-(x+h) - (-x)}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$

When $x = 0$, the situation is a bit more complicated:

$$f'(0) = \lim_{h\to 0} \frac{|0+h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

This is undefined, as $\displaystyle\lim_{h\to 0^+} \frac{|h|}{h} = +1$ and $\displaystyle\lim_{h\to 0^-} \frac{|h|}{h} = -1$, so:

$$\mathbf{D}(|x|) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

or, equivalently, $\mathbf{D}(|x|) = \dfrac{|x|}{x}$. ◀

The derivative of $|x|$ agrees with the function $\text{sgn}(x)$ defined in Chapter 0, except at $x = 0$: $\mathbf{D}(|x|)$ is undefined at $x = 0$ but $\text{sgn}(0) = 0$.

Practice 5. Graph $y = |x - 2|$ and $y = |2x|$ and use the *graphs* to determine $\mathbf{D}(|x - 2|)$ and $\mathbf{D}(|2x|)$.

So far we have emphasized the derivative as the slope of the line tangent to a graph. That very visual interpretation is very useful when examining the graph of a function, and we will continue to use it. Derivatives, however, are employed in a wide variety of fields and applications, and some of these fields use other interpretations. A few commonly used interpretations of the derivative follow.

Interpretations of the Derivative

General

Rate of Change The function $f'(x)$ is the rate of change of the function at x. If the units for x are years and the units for $f(x)$ are people, then the units for $\frac{df}{dx}$ are $\frac{\text{people}}{\text{year}}$, a rate of change in population.

Graphical

Slope $f'(x)$ is the slope of the line tangent to the graph of f at $(x, f(x))$.

Physical

Velocity If $f(x)$ is the position of an object at time x, then $f'(x)$ is the velocity of the object at time x. If the units for x are hours and $f(x)$ is distance, measured in miles, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{miles}}{\text{hour}}$, miles per hour, which is a measure of velocity.

Acceleration If $f(x)$ is the velocity of an object at time x, then $f'(x)$ is the acceleration of the object at time x. If the units for x are hours and $f(x)$ has the units $\frac{\text{miles}}{\text{hour}}$, then the units for the acceleration $f'(x) = \frac{df}{dx}$ are $\frac{\text{miles/hour}}{\text{hour}} = \frac{\text{miles}}{\text{hour}^2}$, "miles per hour per hour."

Magnification $f'(x)$ is the magnification factor of the function f for points close to x. If a and b are two points very close to x, then the distance between $f(a)$ and $f(b)$ will be close to $f'(x)$ times the original distance between a and b: $f(b) - f(a) \approx f'(x)(b - a)$.

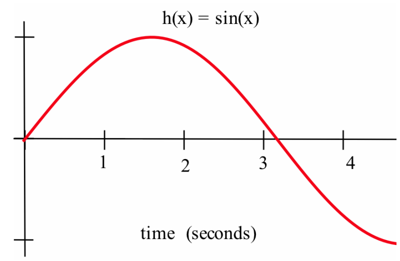
Business

Marginal Cost If $f(x)$ is the total cost of producing x objects, then $f'(x)$ is the marginal cost, at a production level of x: (approximately) the additional cost of making one more object once we have already made x objects. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{dollars}}{\text{bicycle}}$, the cost per bicycle.

Marginal Profit If $f(x)$ is the total profit from producing and selling x objects, then $f'(x)$ is the marginal profit: the profit to be made from producing and selling one more object. If the units for x are bicycles and the units for $f(x)$ are dollars, then the units for $f'(x) = \frac{df}{dx}$ are $\frac{\text{dollars}}{\text{bicycle}}$, the profit per bicycle.

In financial contexts, the word "marginal" usually refers to the derivative or rate of change of some quantity. One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The vocabulary and problems may be different, but the ideas and even the notations of calculus remain useful.

Example 4. A small cork is bobbing up and down, and at time t seconds it is $h(t) = \sin(t)$ feet above the mean water level (see margin). Find the height, velocity and acceleration of the cork when $t = 2$ seconds. (Include the proper units for each answer.)



Solution. $h(t) = \sin(t)$ represents the height of the cork at any time t, so the height of the cork when $t = 2$ is $h(2) = \sin(2) \approx 0.91$ feet above the mean water level.

The velocity is the derivative of the position, so $v(t) = \frac{d}{dt}h(t) = \frac{d}{dt}\sin(t) = \cos(t)$. The derivative of position is the limit of $\frac{\Delta h}{\Delta t}$, so the units are $\frac{\text{feet}}{\text{seconds}}$. After 2 seconds, the velocity is $v(2) = \cos(2) \approx -0.42$ feet per second.

The acceleration is the derivative of the velocity, so $a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\cos(t) = -\sin(t)$. The derivative of velocity is the limit of $\frac{\Delta v}{\Delta t}$, so the units are $\frac{\text{feet/second}}{\text{seconds}}$ or $\frac{\text{feet}}{\text{second}^2}$. After 2 seconds the acceleration is $a(2) = -\sin(2) \approx -0.91\frac{\text{ft}}{\text{sec}^2}$. ◄

Practice 6. Find the height, velocity and acceleration of the cork in the previous example after 1 second.

A Most Useful Formula: $\mathbf{D}(x^n)$

Functions that include powers of x are very common (every polynomial is a sum of terms that include powers of x) and, fortunately, it is easy to calculate the derivatives of such powers. The "pattern" emerging from the first few examples in this section is, in fact, true for all powers of x. We will only state and prove the "pattern" here for positive integer powers of x, but it is also true for other powers (as we will prove later).

Theorem: If n is a positive integer, then: $\mathbf{D}(x^n) = n \cdot x^{n-1}$

This theorem is an example of the power of *generality* and *proof* in mathematics. Rather than resorting to the definition when we encounter a new exponent p in the form x^p (imagine using the definition to calculate the derivative of x^{307}), we can justify the pattern for all positive integer exponents n, and then simply apply the result for whatever exponent we have. We know, from the first examples in this section, that the theorem is true for $n = 1, 2$ and 3, but no number of *examples* would guarantee that the pattern is true for all exponents. We need a proof that what we *think* is true really *is* true.

Proof. With $f(x) = x^n$, $f(x+h) = (x+h)^n$, and in order to simplify $f(x+h) - f(x) = (x+h)^n - x^n$, we will need to expand $(x+h)^n$. However, we really only need to know the first two terms of the expansion and to know that all of the other terms of the expansion contain a power of h of at least 2.

You may also be familiar with Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Among many beautiful and amazing properties, the numbers in row n of the triangle (counting the first row as row 0) give the coefficients in the expansion of $(A+B)^n$. Notice that each entry in the interior of the triangle is the sum of the two numbers immediately above it.

The Binomial Theorem from algebra says (for $n > 3$) that:

$$(x+h)^n = x^n + n \cdot x^{n-1}h + a \cdot x^{n-2}h^2 + b \cdot x^{n-3}h^3 + \cdots + h^n$$

where a and b represent numerical coefficients. (Expand $(x+h)^n$ for a few different values of n to convince yourself of this result.) Then:

$$\mathbf{D}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Now expand $(x+h)^n$ to get:

$$\lim_{h \to 0} \frac{x^n + n \cdot x^{n-1}h + a \cdot x^{n-2}h^2 + b \cdot x^{n-3}h^3 + \cdots + h^n - x^n}{h}$$

Eliminating $x^n - x^n$ we get:

$$\lim_{h \to 0} \frac{n \cdot x^{n-1}h + a \cdot x^{n-2}h^2 + b \cdot x^{n-3}h^3 + \cdots + h^n}{h}$$

and we can then factor h out of the numerator:

$$\lim_{h \to 0} \frac{h(n \cdot x^{n-1} + a \cdot x^{n-2}h + b \cdot x^{n-3}h^2 + \cdots + h^{n-1})}{h}$$

and divide top and bottom by the factor h:

$$\lim_{h \to 0} \left[n \cdot x^{n-1} + a \cdot x^{n-2}h + b \cdot x^{n-3}h^2 + \cdots + h^{n-1} \right]$$

We are left with a polynomial in h and can now compute the limit by simply evaluating the polynomial at $h = 0$ to get $\mathbf{D}(x^n) = n \cdot x^{n-1}$. □

Practice 7. Calculate $\mathbf{D}(x^5)$, $\frac{d}{dx}(x^2)$, $\mathbf{D}(x^{100})$, $\frac{d}{dt}(t^{31})$ and $\mathbf{D}(x^0)$.

We will occasionally use the result of the theorem for the derivatives of **all** constant powers of x even though it has only been proven for positive integer powers, so far. A proof of a more general result (for all rational powers of x) appears in Section 2.9

Example 5. Find $\mathbf{D}\left(\frac{1}{x}\right)$ and $\frac{d}{dx}(\sqrt{x})$.

Solution. Rewriting the fraction using a negative exponent:

$$\mathbf{D}\left(\frac{1}{x}\right) = \mathbf{D}(x^{-1}) = -1 \cdot x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

Rewriting the square root using a fractional exponent:

$$\frac{d}{dx}(\sqrt{x}) = \mathbf{D}(x^{\frac{1}{2}}) = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

These results can also be obtained by using the definition of the derivative, but the algebra involved is slightly awkward. ◀

Practice 8. Find $\mathbf{D}(x^{\frac{3}{2}})$, $\frac{d}{dx}(x^{\frac{1}{3}})$, $\mathbf{D}\left(\frac{1}{\sqrt{x}}\right)$ and $\frac{d}{dt}(t^{\pi})$.

Example 6. It costs $\sqrt{x}$ hundred dollars to run a training program for x employees.

(a) How much does it cost to train 100 employees? 101 employees? If you already need to train 100 employees, how much additional money will it cost to add 1 more employee to those being trained?

(b) For $f(x) = \sqrt{x}$, calculate $f'(x)$ and evaluate f' at $x = 100$. How does $f'(100)$ compare with the last answer in part (a)?

Solution. (a) Put $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ hundred dollars, the cost to train x employees. Then $f(100) = \$1000$ and $f(101) = \$1004.99$, so it costs \$4.99 additional to train the 101st employee. (b) $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ so $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$ hundred dollars = \$5.00. Clearly $f'(100)$ is very close to the actual additional cost of training the 101st employee. ◀

Important Information and Results

This section contains a great deal of important information that we will continue to use throughout the rest of the course.

So it is worthwhile to collect here some of those important ideas.

Definition of Derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Valid if the limit exists and is finite.

Notations for the Derivative: $f'(x)$, $\mathbf{D}(f(x))$, $\frac{df}{dx}$

Tangent Line Equation: $y = f(a) + f'(a) \cdot (x - a)$

An equation of the line tangent to the graph of f at $(a, f(a))$.

Formulas:

- $\mathbf{D}(\text{constant}) = 0$
- $\mathbf{D}(x^n) = n \cdot x^{n-1}$
- $\mathbf{D}(\sin(x)) = \cos(x)$ and $\mathbf{D}(\cos(x)) = -\sin(x)$
- $\mathbf{D}(|x|) = \begin{cases} 1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} = \frac{|x|}{x}$

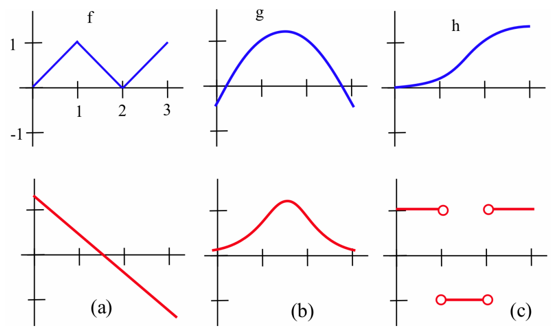
Proved for n = positive integer, but true for all constants n.

Interpretations of $f'(x)$:

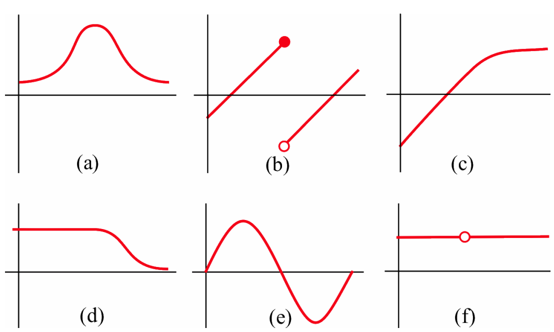
- Slope of a line tangent to a graph
- Instantaneous rate of change of a function at a point
- Velocity or acceleration
- Magnification factor
- Marginal change

2.1 Problems

1. Match the functions f, g and h shown below with the graphs of their derivatives (show in the bottom row).



2. The figure below shows six graphs, three of which are derivatives of the other three. Match the functions with their derivatives.



In Problems 3–6, find the slope m_{sec} of the secant line through the two given points and then calculate $m_{\tan} = \lim\limits_{h \to 0} m_{\text{sec}}$.

3. $f(x) = x^2$
   (a) $(-2,4)$, $(-2+h, (-2+h)^2)$
   (b) $(0.5, 0.25)$, $(0.5+h, (0.5+h)^2)$

4. $f(x) = 3 + x^2$
   (a) $(1,4)$, $(1+h, 3+(1+h)^2)$
   (b) $(x, 3+x^2)$, $(x+h, 3+(x+h)^2)$

5. $f(x) = 7x - x^2$
   (a) $(1,6)$, $(1+h, 7(1+h)-(1+h)^2)$
   (b) $(x, 7x-x^2)$, $(x+h, 7(x+h)-(x+h)^2)$

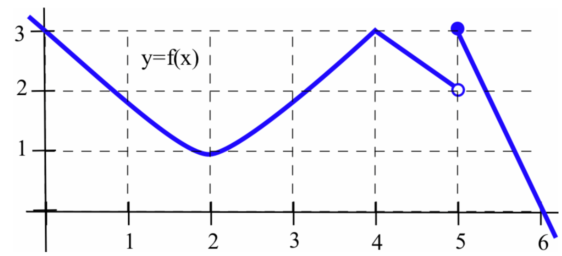
6. $f(x) = x^3 + 4x$
   (a) $(1,5)$, $(1+h, (1+h)^3 + 4(1+h))$
   (b) (x, x^3+4x), $(x+h, (x+h)^3 + 4(x+h))$

7. Use the graph below to estimate the values of these limits. (It helps to recognize what the limit represents.)

   (a) $\lim\limits_{h \to 0} \dfrac{f(0+h)-f(0)}{h}$ (b) $\lim\limits_{h \to 0} \dfrac{f(1+h)-f(1)}{h}$

   (c) $\lim\limits_{w \to 0} \dfrac{f(2+w)-1}{w}$ (d) $\lim\limits_{h \to 0} \dfrac{f(3+h)-f(3)}{h}$

   (e) $\lim\limits_{h \to 0} \dfrac{f(4+h)-f(4)}{h}$ (f) $\lim\limits_{s \to 0} \dfrac{f(5+s)-f(5)}{s}$

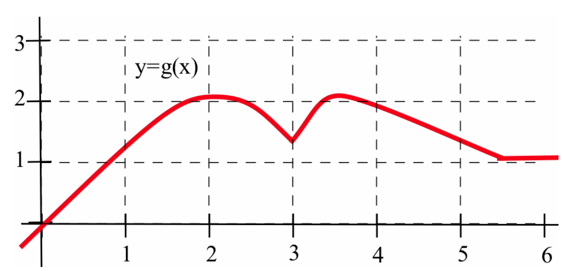


8. Use the graph below to estimate the values of these limits.

   (a) $\lim\limits_{h \to 0} \dfrac{g(0+h)-g(0)}{h}$ (b) $\lim\limits_{h \to 0} \dfrac{g(1+h)-g(1)}{h}$

   (c) $\lim\limits_{w \to 0} \dfrac{g(2+w)-2}{w}$ (d) $\lim\limits_{h \to 0} \dfrac{g(3+h)-g(3)}{h}$

   (e) $\lim\limits_{h \to 0} \dfrac{g(4+h)-g(4)}{h}$ (f) $\lim\limits_{s \to 0} \dfrac{g(5+s)-g(5)}{s}$



In Problems 9–12, use the definition of the derivative to calculate $f'(x)$ and then evaluate $f'(3)$.

9. $f(x) = x^2 + 8$
10. $f(x) = 5x^2 - 2x$
11. $f(x) = 2x^3 - 5x$
12. $f(x) = 7x^3 + x$

13. Graph $f(x) = x^2$, $g(x) = x^2 + 3$ and $h(x) = x^2 - 5$. Calculate the derivatives of f, g and h.

14. Graph $f(x) = 5x$, $g(x) = 5x + 2$ and $h(x) = 5x - 7$. Calculate the derivatives of f, g and h.

In Problems 15–18, find the slopes and equations of the lines tangent to $y = f(x)$ at the given points.

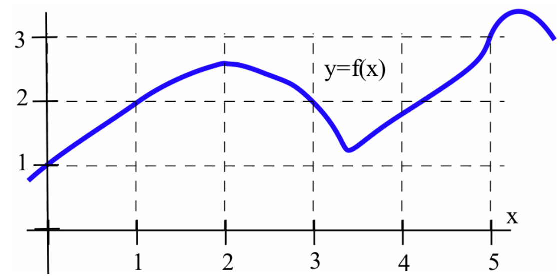
15. $f(x) = x^2 + 8$ at $(1, 9)$ and $(-2, 12)$.
16. $f(x) = 5x^2 - 2x$ at $(2, 16)$ and $(0, 0)$.
17. $f(x) = \sin(x)$ at $(\pi, 0)$ and $(\frac{\pi}{2}, 1)$.
18. $f(x) = |x + 3|$ at $(0, 3)$ and $(-3, 0)$.

19. (a) Find an equation of the line tangent to the graph of $y = x^2 + 1$ at the point $(2, 5)$.
    (b) Find an equation of the line perpendicular to the graph of $y = x^2 + 1$ at $(2, 5)$.
    (c) Where is the line tangent to the graph of $y = x^2 + 1$ horizontal?
    (d) Find an equation of the line tangent to the graph of $y = x^2 + 1$ at the point (p, q).
    (e) Find the point(s) (p, q) on the graph of $y = x^2 + 1$ so the tangent line to the curve at (p, q) goes through the point $(1, -7)$.

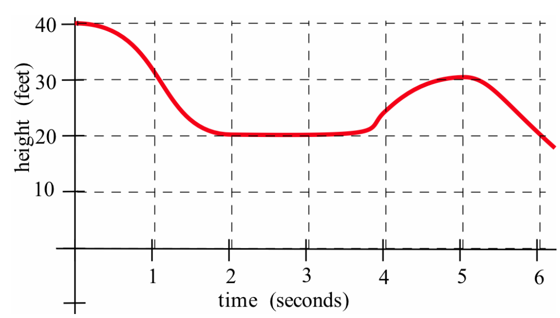
20. (a) Find an equation of the line tangent to the graph of $y = x^3$ at the point $(2, 8)$.
    (b) Where, if ever, is the line tangent to the graph of $y = x^3$ horizontal?
    (c) Find an equation of the line tangent to the graph of $y = x^3$ at the point (p, q).
    (d) Find the point(s) (p, q) on the graph of $y = x^3$ so the tangent line to the curve at (p, q) goes through the point $(16, 0)$.

21. (a) Find the angle that the line tangent to $y = x^2$ at $(1, 1)$ makes with the x-axis.
    (b) Find the angle that the line tangent to $y = x^3$ at $(1, 1)$ makes with the x-axis.
    (c) The curves $y = x^2$ and $y = x^3$ intersect at the point $(1, 1)$. Find the angle of intersection of the two curves (actually the angle between their tangent lines) at the point $(1, 1)$.

22. The figure below shows the graph of $y = f(x)$. Sketch a graph of $y = f'(x)$.



23. The figure below shows the graph of the height of an object at time t. Sketch a graph of the object's upward velocity. What are the units for each axis on the velocity graph?



24. Fill in the table with units for $f'(x)$.

units for x	units for $f(x)$	units for $f'(x)$
hours	miles	
people	automobiles	
dollars	pancakes	
days	trout	
seconds	miles per second	
seconds	gallons	
study hours	test points	

25. A rock dropped into a deep hole will drop $d(x) = 16x^2$ feet in x seconds.

   (a) How far into the hole will the rock be after 4 seconds? After 5 seconds?

   (b) How fast will it be falling at exactly 4 seconds? After 5 seconds? After x seconds?

26. It takes $T(x) = x^2$ hours to weave x small rugs. What is the marginal production time to weave a rug? (Be sure to include the units with your answer.)

27. It costs $C(x) = \sqrt{x}$ dollars to produce x golf balls. What is the marginal production cost to make a golf ball? What is the marginal production cost when $x = 25$? When $x = 100$? (Include units.)

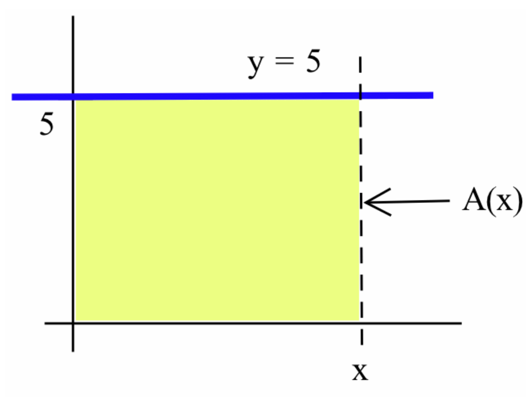
28. Define $A(x)$ to be the **area** bounded by the t- and y-axes, the line $y = 5$ and a vertical line at $t = x$ (see figure below).

   (a) Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.

   (b) Find a formula for $A(x)$ valid for $x \geq 0$.

   (c) Determine $A'(x)$.

   (d) What does $A'(x)$ represent?



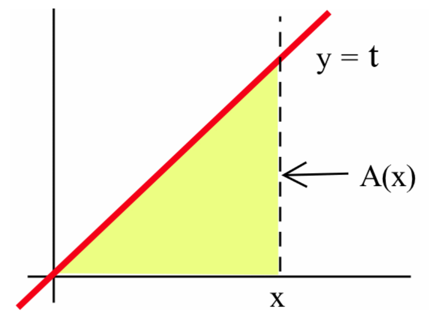
29. Define $A(x)$ to be the **area** bounded by the t-axis, the line $y = t$, and a vertical line at $t = x$ (see figure below).

   (a) Evaluate $A(0)$, $A(1)$, $A(2)$ and $A(3)$.

   (b) Find a formula for $A(x)$ valid for $x \geq 0$.

   (c) Determine $A'(x)$.

   (d) What does $A'(x)$ represent?



30. Compute each derivative.

   (a) $\mathbf{D}(x^{12})$

   (b) $\frac{d}{dx}(\sqrt[7]{x})$

   (c) $\mathbf{D}\left(\frac{1}{x^3}\right)$

   (d) $\frac{d}{dx}(x^c)$

   (e) $\mathbf{D}(|x-2|)$

31. Compute each derivative.

   (a) $\mathbf{D}(x^9)$

   (b) $\frac{d}{dx}(x^{\frac{2}{3}})$

   (c) $\mathbf{D}\left(\frac{1}{x^4}\right)$

   (d) $\frac{d}{dx}(x^\pi)$

   (e) $\mathbf{D}(|x+5|)$

In Problems 32–37, find a function f that has the given derivative. (Each problem has several correct answers, just find one of them.)

32. $f'(x) = 4x + 3$

33. $f'(x) = 3x^2 + 8x$

34. $\mathbf{D}(f(x)) = 12x^2 - 7$

35. $f'(t) = 5\cos(t)$

36. $\frac{d}{dx}f(x) = 2x - \sin(x)$

37. $\mathbf{D}(f(x)) = x + x^2$

2.1 Practice Answers

1. The graph of $f(x) = 7x$ is a line through the origin. The slope of the line is 7. For all x:

$$m_{\tan} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{7(x+h)-7x}{h} = \lim_{h\to 0} \frac{7h}{h} = \lim_{h\to 0} 7 = 7$$

2. $f(x) = x^3 \Rightarrow f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$ so:

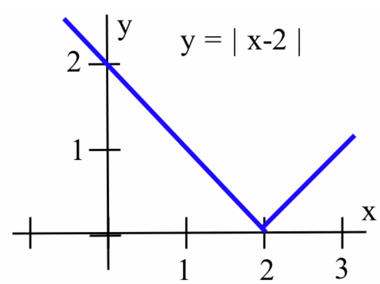
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2 \end{aligned}$$

At the point $(2,8)$, the slope of the tangent line is $3(2)^2 = 12$ so an equation of the tangent line is $y - 8 = 12(x-2)$ or $y = 12x - 16$.

3. $\mathbf{D}(x^4) = 4x^3$, $\mathbf{D}(x^5) = 5x^4$, $\mathbf{D}(x^{43}) = 43x^{42}$,
$\mathbf{D}(\sqrt{x}) = \mathbf{D}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, $\mathbf{D}(x^\pi) = \pi x^{\pi-1}$

4. Proceeding as we did to find the derivative to $\sin(x)$:

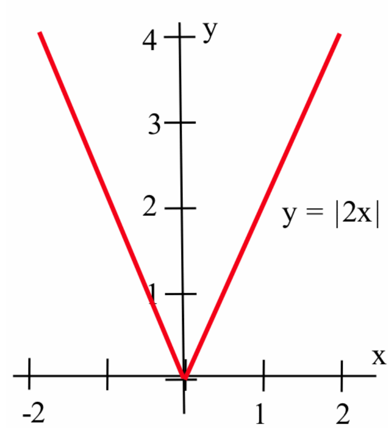
$$\begin{aligned} \mathbf{D}(\cos(x)) &= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} = \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-cos(x)}{h} \\ &= \lim_{h\to 0} \left[\cos(x)\cdot\frac{\cos(h)-1}{h} - \sin(x)\frac{\sin(h)}{h}\right] = \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x) \end{aligned}$$



5. See margin figure for the graphs of $y = |x-2|$ and $y = |2x|$.

$$\mathbf{D}(|x-2|) = \begin{cases} 1 & \text{if } x > 2 \\ \text{undefined} & \text{if } x = 2 \\ -1 & \text{if } x < 2 \end{cases} = \frac{|x-2|}{x-2}$$

$$\mathbf{D}(|2x|) = \begin{cases} 2 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -2 & \text{if } x < 0 \end{cases} = \frac{2\,|x|}{x}$$



6. $h(t) = \sin(t)$ so $h(1) = \sin(1) \approx 0.84$ ft;
$v(t) = \cos(t)$ so $v(1) = \cos(1) \approx 0.54$ ft/sec;
$a(t) = -\sin(t)$ so $a(1) = -\sin(1) \approx -0.84$ ft/sec^2.

7. $\mathbf{D}(x^5) = 5x^4$, $\frac{d}{dx}(x^2) = 2x^1 = 2x$, $\mathbf{D}(x^{100}) = 100x^{99}$, $\frac{d}{dt}(t^{31}) = 31t^{30}$ and $\mathbf{D}(x^0) = 0x^{-1} = 0$ or $\mathbf{D}(x^0) = \mathbf{D}(1) = 0$

8. $\mathbf{D}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}}$, $\frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{-\frac{2}{3}}$, $\mathbf{D}(\frac{1}{\sqrt{x}}) = \mathbf{D}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{3}{2}}$, $\frac{d}{dt}(t^\pi) = \pi t^{\pi-1}$.

2.2 *Derivatives: Properties and Formulas*

This section begins with a look at which functions have derivatives. Then we'll examine how to calculate derivatives of elementary combinations of basic functions. By knowing the derivatives of some basic functions and just a few differentiation patterns, you will be able to calculate the derivatives of a tremendous variety of functions. This section contains most — but not quite all — of the general differentiation patterns you will ever need.

Which Functions Have Derivatives?

A function must be continuous in order to be differentiable.

It is vital to understand what this theorem tells us and what it does **not** tell us: If a function is differentiable at a point, then the function is automatically continuous there. If the function is continuous at a point, then the function may or may not be differentiable there.

> **Theorem**:
>
> If a function is differentiable at a point
> then it is continuous at that point.

Proof. Assume that the hypothesis (f is differentiable at the point c) is true. Then $\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ must exist and be equal to $f'(c)$. We want to show that f must necessarily be continuous at c, so we need to show that $\lim\limits_{h\to 0} f(c+h) = f(c)$.

It's not yet obvious why we want to do so, but we can write:

$$f(c+h) = f(c) + \frac{f(c+h)-f(c)}{h} \cdot h$$

and then compute the limit of both sides of this expression:

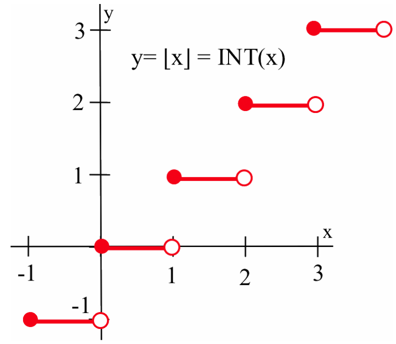
$$\begin{aligned}\lim_{h\to 0} f(c+h) &= \lim_{h\to 0} \left(f(c) + \frac{f(c+h)-f(c)}{h} \cdot h \right) \\ &= \lim_{h\to 0} f(c) + \lim_{h\to 0} \left(\frac{f(c+h)-f(c)}{h} \cdot h \right) \\ &= \lim_{h\to 0} f(c) + \lim_{h\to 0} \left(\frac{f(c+h)-f(c)}{h} \right) \cdot \lim_{h\to 0} h \\ &= f(c) + f'(c) \cdot 0 = f(c)\end{aligned}$$

Therefore f is continuous at c. □

We often use the contrapositive form of this theorem, which tells us about some functions that do **not** have derivatives.

> **Contrapositive Form of the Theorem**:
>
> If f is not continuous at a point
> then f is not differentiable at that point.

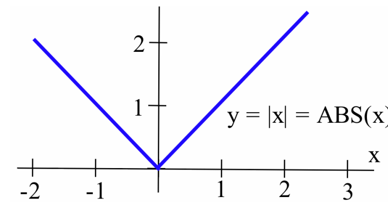
Example 1. Show that $f(x) = \lfloor x \rfloor$ is not continuous and not differentiable at $x = 2$ (see margin figure).



Solution. The one-sided limits $\lim_{x\to 2^+} \lfloor x \rfloor = 2$ and $\lim_{x\to 2^-} \lfloor x \rfloor = 1$ have different values, so $\lim_{x\to 2} \lfloor x \rfloor$ does not exist. Therefore $f(x) = \lfloor x \rfloor$ is not continuous at 2, and as a result it is not differentiable at 2. ◀

Lack of continuity implies lack of differentiability, but the next examples show that continuity is **not** enough to guarantee differentiability.

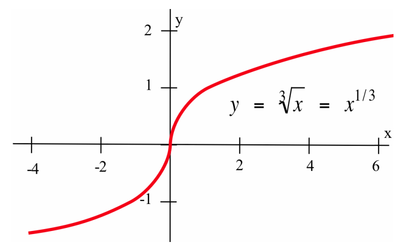
Example 2. Show that $f(x) = |x|$ is continuous but **not** differentiable at $x = 0$ (see margin figure).



Solution. We know that $\lim_{x\to 0} |x| = 0 = |0|$, so f is continuous at 0, but in Section 2.1 we saw that $|x|$ was not differentiable at $x = 0$. ◀

A function is not differentiable at a cusp or a "corner."

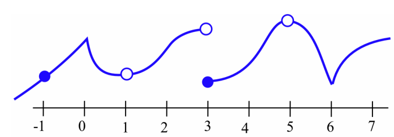
Example 3. Show that $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ is continuous but **not** differentiable at $x = 0$ (see margin figure).



Solution. We can verify that $\lim_{x\to 0^+} \sqrt[3]{x} = \lim_{x\to 0^-} \sqrt[3]{x} = 0$, so $\lim_{x\to 0} \sqrt[3]{x} = 0 = \sqrt[3]{0}$ so f is continuous at 0. But $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$, which is undefined at $x = 0$, so f is not differentiable at 0. ◀

A function is not differentiable where its tangent line is vertical.

Practice 1. At which integer values of x is the graph of f in the margin figure continuous? Differentiable?



Graphically, a function is **continuous** if and only if its graph is "connected" and does not have any holes or breaks. Graphically, a function is **differentiable** if and only if it is continuous and its graph is "smooth" with no corners or vertical tangent lines.

Derivatives of Elementary Combinations of Functions

We now begin to compute derivatives of more complicated functions built from combinations of simpler functions.

Example 4. The derivative of $f(x) = x$ is $\mathbf{D}(f(x)) = 1$ and the derivative of $g(x) = 5$ is $\mathbf{D}(g(x)) = 0$. What are the derivatives of the elementary combinations: $3 \cdot f$, $f + g$, $f - g$, $f \cdot g$ and $\frac{f}{g}$?

Solution. The first three derivatives follow "nice" patterns:

$$\begin{aligned} \mathbf{D}(3 \cdot f(x)) &= \mathbf{D}(3x) = 3 = 3 \cdot 1 = 3 \cdot \mathbf{D}(f(x)) \\ \mathbf{D}(f(x) + g(x)) &= \mathbf{D}(x+5) = 1 = 1 + 0 = \mathbf{D}(f(x)) + \mathbf{D}(g(x)) \\ \mathbf{D}(f(x) - g(x)) &= \mathbf{D}(x-5) = 1 = 1 - 0 = \mathbf{D}(f(x)) - \mathbf{D}(g(x)) \end{aligned}$$

yet the other two derivatives fail to follow the same "nice" patterns: $\mathbf{D}(f(x) \cdot g(x)) = \mathbf{D}(5x) = 5$ but $\mathbf{D}(f(x)) \cdot \mathbf{D}(g(x)) = 1 \cdot 0 = 0$, and $\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \mathbf{D}\left(\frac{x}{5}\right) = \frac{1}{5}$ but $\frac{\mathbf{D}(f(x))}{\mathbf{D}(g(x))} = \frac{1}{0}$ is undefined. ◀

The two very simple functions in the previous example show that, in general, $\mathbf{D}(f \cdot g) \neq \mathbf{D}(f) \cdot \mathbf{D}(g)$ and $\mathbf{D}\left(\frac{f}{g}\right) \neq \frac{\mathbf{D}(f)}{\mathbf{D}(g)}$.

Practice 2. For $f(x) = 6x + 8$ and $g(x) = 2$, compute the derivatives of $3 \cdot f$, $f + g$, $f - g$, $f \cdot g$ and $\frac{f}{g}$.

Main Differentiation Theorem:

If f and g are differentiable at x, then:

(a) **Constant Multiple Rule**:

$$\mathbf{D}(k \cdot f(x)) = k \cdot \mathbf{D}(f(x))$$

(b) **Sum Rule**:

$$\mathbf{D}(f(x) + g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x))$$

(c) **Difference Rule**:

$$\mathbf{D}(f(x) - g(x)) = \mathbf{D}(f(x)) - \mathbf{D}(g(x))$$

(d) **Product Rule**:

$$\mathbf{D}(f(x) \cdot g(x)) = f(x) \cdot \mathbf{D}(g(x)) + g(x) \cdot \mathbf{D}(f(x))$$

(e) **Quotient Rule**:

$$\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \mathbf{D}(f(x)) - f(x) \cdot \mathbf{D}(g(x))}{[g(x)]^2}$$

Part (e) requires that $g(x) \neq 0$.

This theorem says that the simple patterns in the previous example for constant multiples of functions and sums and differences of functions are true for all differentiable functions. It also includes the correct patterns for derivatives of products and quotients of differentiable functions.

The proofs of parts (a), (b) and (c) of this theorem are straightforward, but parts (d) and (e) require some clever algebraic manipulations. Let's look at some examples before tackling the proof.

Example 5. Recall that $\mathbf{D}(x^2) = 2x$ and $\mathbf{D}(\sin(x)) = \cos(x)$. Find $\mathbf{D}(3\sin(x))$ and $\frac{d}{dx}(5x^2 - 7\sin(x))$.

Solution. Computing $\mathbf{D}(3\sin(x))$ requires part (a) of the theorem with $k = 3$ and $f(x) = \sin(x)$ so $\mathbf{D}(3 \cdot \sin(x)) = 3 \cdot \mathbf{D}(\sin(x)) = 3\cos(x)$, while $\dfrac{d}{dx}(5x^2 - 7\sin(x))$ uses part (c) of the theorem with $f(x) = 5x^2$ and $g(x) = 7\sin(x)$ so:

$$\begin{aligned}\frac{d}{dx}(5x^2 - 7\sin(x)) &= \frac{d}{dx}(5x^2) - \frac{d}{dx}(7\sin(x)) \\ &= 5 \cdot \frac{d}{dx}(x^2) - 7 \cdot \frac{d}{dx}(\sin(x)) \\ &= 5(2x) - 7(\cos(x))\end{aligned}$$

which simplifies to $10x - 7\cos(x)$. ◀

Practice 3. Find $\mathbf{D}(x^3 - 5\sin(x))$ and $\dfrac{d}{dx}(\sin(x) - 4x^3)$.

Practice 4. The table below gives the values of functions f and g, as well as their derivatives, at various points. Fill in the missing values for $\mathbf{D}(3 \cdot f(x))$, $\mathbf{D}(2 \cdot f(x) + g(x))$ and $\mathbf{D}(3 \cdot g(x) - f(x))$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\mathbf{D}(3f(x))$	$\mathbf{D}(2f(x) + g(x))$	$\mathbf{D}(3g(x) - f(x))$
0	3	−2	−4	3			
1	2	−1	1	0			
2	4	2	3	1			

Practice 5. Use the Main Differentiation Theorem to complete the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\mathbf{D}(f(x) \cdot g(x))$	$\mathbf{D}\left(\frac{f(x)}{g(x)}\right)$	$\mathbf{D}\left(\frac{g(x)}{f(x)}\right)$
0	3	−2	−4	3			
1	2	−1	1	0			
2	4	2	3	1			

Example 6. Determine $\mathbf{D}(x^2 \cdot \sin(x))$ and $\dfrac{d}{dx}\left(\dfrac{x^3}{\sin(x)}\right)$.

Solution. (a) Use the Product Rule with $f(x) = x^2$ and $g(x) = \sin(x)$:

$$\begin{aligned}\mathbf{D}(x^2 \cdot \sin(x)) &= \mathbf{D}(f(x) \cdot g(x)) = f(x) \cdot \mathbf{D}(g(x)) + g(x) \cdot \mathbf{D}(f(x)) \\ &= x^2 \cdot \mathbf{D}(\sin(x)) + \sin(x) \cdot \mathbf{D}(x^2) \\ &= x^2 \cdot \cos(x) + \sin(x) \cdot 2x = x^2\cos(x) + 2x\sin(x)\end{aligned}$$

Many calculus students find it easier to remember the Product Rule in words: "the first function times the derivative of the second plus the second function times the derivative of the first."

(b) Use the Quotient Rule with $f(x) = x^3$ and $g(x) = \sin(x)$:

The Quotient Rule in words: "the bottom times the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared."

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^3}{\sin(x)}\right) &= \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \\ &= \frac{g(x)\cdot \mathbf{D}(f(x)) - f(x)\cdot \mathbf{D}(g(x))}{[g(x)]^2} \\ &= \frac{\sin(x)\cdot \mathbf{D}(x^3) - x^3\cdot \mathbf{D}(\sin(x))}{[\sin(x)]^2} \\ &= \frac{\sin(x)\cdot 3x^2 - x^3\cdot\cos(x)}{\sin^2(x)} \\ &= \frac{3x^2\sin(x) - x^3\cdot\cos(x)}{\sin^2(x)}\end{aligned}$$

which could also be rewritten in terms of $\csc(x)$ and $\cot(x)$. ◀

Practice 6. Find $\mathbf{D}((x^2+1)(7x-3))$, $\frac{d}{dt}\left(\frac{3t-2}{5t+1}\right)$ and $\mathbf{D}\left(\frac{\cos(x)}{x}\right)$.

Now that we've seen how to use the theorem, let's prove it.

Proof. The only general fact we have about derivatives is the definition as a limit, so our proofs here will need to recast derivatives as limits and then use some results about limits. The proofs involve applications of the definition of the derivative and results about limits.

(a) Using the derivative definition and the limit laws:

$$\begin{aligned}\mathbf{D}(k\cdot f(x)) &= \lim_{h\to 0}\frac{k\cdot f(x+h) - k\cdot f(x)}{h} \\ &= \lim_{h\to 0} k\cdot\frac{f(x+h)-f(x)}{h} \\ &= k\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = k\cdot\mathbf{D}(f(x))\end{aligned}$$

(b) You try it (see Practice problem that follows).

(c) Once again using the derivative definition and the limit laws:

$$\begin{aligned}\mathbf{D}(f(x)-g(x)) &= \lim_{h\to 0}\frac{[f(x+h)-g(x+h)]-[f(x)-g(x)]}{h} \\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]-[g(x+h)-g(x)]}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \mathbf{D}(f(x)) - \mathbf{D}(g(x))\end{aligned}$$

The proofs of parts (d) and (e) of the theorem are more complicated but only involve elementary techniques, used in just the right way.

Sometimes we will omit such computational proofs, but the Product and Quotient Rules are fundamental techniques you will need hundreds of times.

(d) By the hypothesis, f and g are differentiable, so:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

and:

$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x)$$

Also, both f and g are continuous (why?) so $\lim_{h\to 0} f(x+h) = f(x)$ and $\lim_{h\to 0} g(x+h) = g(x)$.

Let $P(x) = f(x)\cdot g(x)$. Then $P(x+h) = f(x+h)\cdot g(x+h)$ and:

$$\begin{aligned}\mathbf{D}(f(x)\cdot g(x)) = \mathbf{D}(P(x)) &= \lim_{h\to 0} \frac{P(x+h)-P(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}\end{aligned}$$

At this stage we need to use some cleverness to add and subtract $f(x)\cdot g(x+h)$ from the numerator (you'll see why shortly):

$$\lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) + [-f(x)\cdot g(x+h) + f(x)\cdot g(x+h)] - f(x)g(x)}{h}$$

We can then split this giant fraction into two more manageable limits:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

and then factor out a common factor from each numerator:

$$\lim_{h\to 0} g(x+h)\cdot\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x)\cdot\frac{g(x+h)-g(x)}{h}$$

Taking limits of each piece (and using the continuity of $g(x)$) we get:

$$\mathbf{D}(f(x)\cdot g(x)) = g(x)\cdot f'(x) + f(x)\cdot g'(x) = g\cdot\mathbf{D}(f) + f\cdot\mathbf{D}(g)$$

The steps for a proof of the Quotient Rule appear in Problem 69. ☐

Practice 7. Prove the Sum Rule: $\mathbf{D}(f(x)+g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x))$. (Refer to the proof of part (c) for guidance.)

Using the Differentiation Rules

You definitely need to memorize the differentiation rules, but it is vitally important that you also know **how** to use them. Sometimes it is clear that the function we want to differentiate is a sum or product of two obvious functions, but we commonly need to differentiate functions that involve several operations and functions. Memorizing the differentiation rules is only the first step in learning to use them.

Example 7. Calculate $\mathbf{D}(x^5 + x \cdot \sin(x))$.

Solution. This function is more difficult because it involves both an addition and a multiplication. Which rule(s) should we use — or, more importantly, which rule should we use **first**?

First apply the Sum Rule to trade one derivative for two easier ones:

$$\begin{aligned}\mathbf{D}(x^5 + x \cdot \sin(x)) &= \mathbf{D}(x^5) + \mathbf{D}(x \cdot \sin(x)) \\ &= 5x^4 + [x \cdot \mathbf{D}(\sin(x)) + \sin(x) \cdot \mathbf{D}(x)] \\ &= 5x^4 + x \cdot \cos(x) + \sin(x)\end{aligned}$$

This last expression involves no more derivatives, so we are done. ◀

If instead of computing the derivative you were evaluating the function $x^5 + x\sin(x)$ for some particular value of x, you would:

- raise x to the 5th power
- calculate $\sin(x)$
- multiply $\sin(x)$ by x and, finally,
- **add** (sum) the values of x^5 and $x\sin(x)$

Notice that the **final** step of your **evaluation** of f indicates the **first** rule to use to calculate the **derivative** of f.

Practice 8. Which differentiation rule should you apply **first** for each of the following?

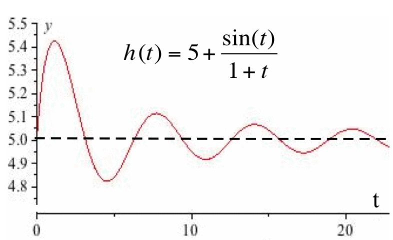
(a) $x \cdot \cos(x) - x^3 \cdot \sin(x)$

(b) $(2x - 3)\cos(x)$

(c) $2\cos(x) - 7x^2$

(d) $\dfrac{\cos(x) + 3x}{\sqrt{x}}$

Practice 9. Calculate $\mathbf{D}\left(\dfrac{x^2 - 5}{\sin(x)}\right)$ and $\dfrac{d}{dt}\left(\dfrac{t^2 - 5}{t \cdot \sin(t)}\right)$.



Example 8. A mass attached to a spring oscillates up and down but the motion becomes "damped" due to friction and air resistance. The height of the mass after t seconds is given by $h(t) = 5 + \dfrac{\sin(t)}{1+t}$ (in feet). Find the height and velocity of the mass after 2 seconds.

Solution. The height is $h(2) = 5 + \dfrac{\sin(2)}{1+2} \approx 5 + \dfrac{0.909}{3} = 5.303$ feet above the ground. The velocity is $h'(2)$, so we must first compute $h'(t)$ and then evaluate the derivative at time $t = 2$:

$$h'(t) = \frac{(1+t) \cdot \cos(t) - \sin(t) \cdot 1}{(1+t)^2}$$

so $h'(2) = \dfrac{3\cos(2) - \sin(2)}{9} \approx \dfrac{-2.158}{9} \approx -0.24$ feet per second. ◀

Practice 10. What are the height and velocity of the weight in the previous example after 5 seconds? What are the height and velocity of the weight be after a "long time" has passed?

Example 9. Calculate $\mathbf{D}(x \cdot \sin(x) \cdot \cos(x))$.

Solution. Clearly we need to use the Product Rule, because the only operation in this function is multiplication. But the Product Rule deals with a product of **two** functions and here we have the product of three: x and $\sin(x)$ and $\cos(x)$. If, however, we think of our two functions as $f(x) = x \cdot \sin(x)$ and $g(x) = \cos(x)$, then we do have the product of two functions and:

$$\begin{aligned}\mathbf{D}(x \cdot \sin(x) \cdot \cos(x)) &= \mathbf{D}(f(x) \cdot g(x)) \\ &= f(x) \cdot \mathbf{D}(g(x)) + g(x) \cdot \mathbf{D}(f(x)) \\ &= x\sin(x) \cdot \mathbf{D}(\cos(x)) + \cos(x) \cdot \mathbf{D}(x\sin(x))\end{aligned}$$

We are not done, but we have traded one hard derivative for two easier ones. We know that $\mathbf{D}(\cos(x)) = -\sin(x)$ and we can use the Product Rule (again) to calculate $\mathbf{D}(x\sin(x))$. Then the last line of our calculation above becomes:

$$x\sin(x) \cdot [-\sin(x)] + \cos(x) \cdot [x\,\mathbf{D}(\sin(x)) + \sin(x)\,\mathbf{D}(x)]$$

and then:

$$-x\sin^2(x) + \cos(x)\,[x\cos(x) + \sin(x)(1)]$$

which simplifies to $-x\sin^2(x) + x\cos^2(x) + \cos(x)\sin(x)$. ◀

Evaluating a Derivative at a Point

The derivative of a function $f(x)$ is a new **function** $f'(x)$ that tells us the slope of the line tangent to the graph of f at each point x. To find the slope of the tangent line at a particular point $(c, f(c))$ on the graph of f, we should *first* calculate the derivative $f'(x)$ and *then* evaluate the function $f'(x)$ at the point $x = c$ to get the **number** $f'(c)$. If you mistakenly evaluate f first, you get a number $f(c)$, and the derivative of a constant is always equal to 0.

Example 10. Determine the slope of the line tangent to the graph of $f(x) = 3x + \sin(x)$ at $(0, f(0))$ and $(1, f(1))$.

Solution. $f'(x) = \mathbf{D}(3x + \sin(x)) = \mathbf{D}(3x) + \mathbf{D}(\sin(x)) = 3 + \cos(x)$. When $x = 0$, the graph of $y = 3x + \sin(x)$ goes through the point $(0, 3(0) + \sin(0)) = (0, 0)$ with slope $f'(0) = 3 + \cos(0) = 4$. When $x = 1$, the graph goes through the point $(1, 3(1) + \sin(1)) \approx (1, 3.84)$ with slope $f'(1) = 3 + \cos(1) \approx 3.54$. ◀

Practice 11. Where do $f(x) = x^2 - 10x + 3$ and $g(x) = x^3 - 12x$ have horizontal tangent lines?

Important Information and Results

This section, like the last one, contains a great deal of important information that we will continue to use throughout the rest of the course, so we collect here some of those important results.

Differentiability and Continuity: If a function is differentiable then it must be continuous. If a function is not continuous then it cannot be differentiable. A function may be continuous at a point and not differentiable there.

Graphically: *Continuous* means "connected"; *differentiable* means "continuous, smooth and not vertical."

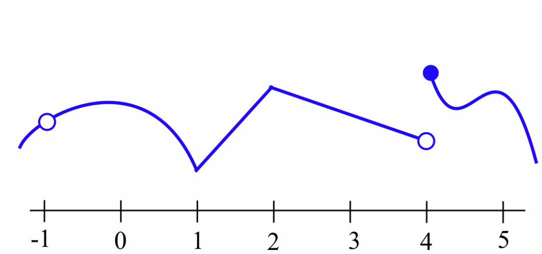
Differentiation Patterns:

- $[k \cdot f(x)]' = k \cdot f'(x)$
- $[f(x) + g(x)]' = f'(x) + g'(x)$
- $[f(x) - g(x)]' = f'(x) - g'(x)$
- $[f(x) \cdot g(x)]' = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
- $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
- The *final step* used to evaluate a function f indicates the *first rule* used to differentiate f.

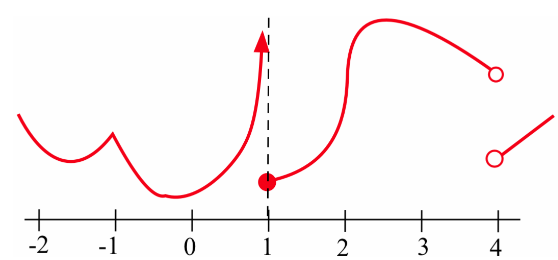
Evaluating a derivative at a point: First differentiate and *then* evaluate.

2.2 *Problems*

1. Use the graph of $y = f(x)$ below to determine:
   (a) at which integers f is continuous.
   (b) at which integers f is differentiable.



2. Use the graph of $y = g(x)$ below to determine:
   (a) at which integers g is continuous.
   (b) at which integers g is differentiable.



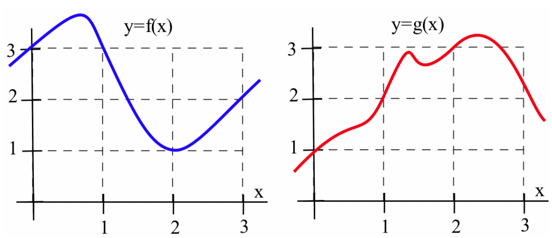
3. Use the values given in the table to determine the values of $f \cdot g$, $\mathbf{D}(f \cdot g)$, $\frac{f}{g}$ and $\mathbf{D}\left(\frac{f}{g}\right)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$f(x) \cdot g(x)$	$\mathbf{D}(f(x) \cdot g(x))$	$\frac{f(x)}{g(x)}$	$\mathbf{D}\left(\frac{f(x)}{g(x)}\right)$
0	2	3	1	5				
1	-3	2	5	-2				
2	0	-3	2	4				
3	1	-1	0	3				

4. Use the values given in the table to determine the values of $f \cdot g$, $\mathbf{D}(f \cdot g)$, $\frac{f}{g}$ and $\mathbf{D}\left(\frac{f}{g}\right)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$f(x) \cdot g(x)$	$\mathbf{D}(f(x) \cdot g(x))$	$\frac{f(x)}{g(x)}$	$\mathbf{D}\left(\frac{f(x)}{g(x)}\right)$
0	4	2	3	−3				
1	0	3	2	1				
2	−2	5	0	−1				
3	−1	−2	−3	4				

5. Use the information in the figure below to plot the values of the functions $f+g$, $f \cdot g$ and $\frac{f}{g}$ and their derivatives at $x = 1, 2$ and 3.



6. Use the information in the figure above to plot the values of the functions $2f$, $f-g$ and $\frac{g}{f}$ and their derivatives at $x = 1, 2$ and 3.

7. Calculate $\mathbf{D}((x-5)(3x+7))$ by:

(a) using the Product Rule.

(b) expanding and then differentiating.

Verify that both methods give the same result.

8. Calculate $\mathbf{D}\left(\frac{x^3-3x+2}{\sqrt{x}}\right)$ by:

(a) using the Quotient Rule.

(b) rewriting and then differentiating.

Verify that both methods give the same result.

In Problems 9–26, compute each derivative.

9. $\frac{d}{dx}\left(19x^3-7\right)$

10. $\frac{d}{dt}\left(5\cos(t)+\frac{\pi}{2}\right)$

11. $\mathbf{D}(\sin(x)+\cos(x))$

12. $\mathbf{D}(7\sin(x)-3\cos(x))$

13. $\mathbf{D}(x^2 \cdot \cos(x))$

14. $\mathbf{D}(\sqrt{x} \cdot \sin(x))$

15. $\mathbf{D}(\sin^2(x))$

16. $\mathbf{D}(\cos^2(x))$

17. $\frac{d}{dx}\left(\frac{\cos(x)}{x^2}\right)$

18. $\frac{d}{dt}\left(\frac{\sin(t)}{t^3}\right)$

19. $\frac{d}{dx}\left(\frac{1}{1+x^2}\right)$

20. $\frac{d}{dt}\left(\frac{t}{1+t^3}\right)$

21. $\frac{d}{d\theta}\left(\frac{1}{\cos(\theta)}\right)$

22. $\frac{d}{d\theta}\left(\frac{1}{\sin(\theta)}\right)$

23. $\frac{d}{d\theta}\left(\frac{\sin(\theta)}{\cos(\theta)}\right)$

24. $\frac{d}{d\theta}\left(\frac{\cos(\theta)}{\sin(\theta)}\right)$

25. $\mathbf{D}\left(8x^5-3x^4+2x^3+7x^2-12x+147\right)$

26. (a) $\mathbf{D}(\sin(x))$ (b) $\mathbf{D}(\sin(x)+7)$
(c) $\mathbf{D}(\sin(x)-8000)$ (d) $\mathbf{D}(\sin(x)+k)$

27. Find values for the constants a, b and c so that the parabola $f(x) = ax^2+bx+c$ has $f(0) = 0$, $f'(0) = 0$ and $f'(10) = 30$.

28. If f is a differentiable function, how are the:

(a) graphs of $y = f(x)$ and $y = f(x)+k$ related?

(b) derivatives of $f(x)$ and $f(x)+k$ related?

29. If f and g are differentiable functions that always differ by a constant ($f(x)-g(x) = k$ for all x) then what can you conclude about their graphs? Their derivatives?

30. If f and g are differentiable functions whose sum is a constant ($f(x)+g(x) = k$ for all x) then what can you conclude about their graphs? Their derivatives?

31. If the product of f and g is a constant (that is, $f(x) \cdot g(x) = k$ for all x) then how are $\frac{\mathbf{D}(f(x))}{f(x)}$ and $\frac{\mathbf{D}(g(x))}{g(x)}$ related?

32. If the quotient of f and g is a constant ($\frac{f(x)}{g(x)} = k$ for all x) then how are $g \cdot f'$ and $f \cdot g'$ related?

In Problems 33–40:

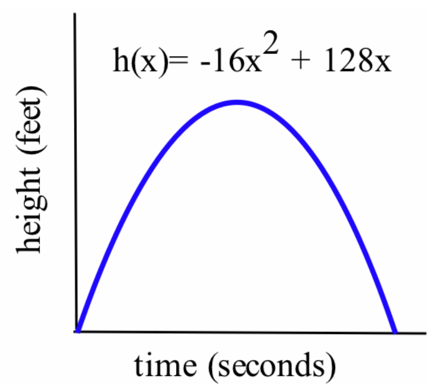
(a) calculate $f'(1)$

(b) determine where $f'(x) = 0$.

33. $f(x) = x^2 - 5x + 13$
34. $f(x) = 5x^2 - 40x + 73$
35. $f(x) = 3x - 2\cos(x)$
36. $f(x) = |x + 2|$
37. $f(x) = x^3 + 9x^2 + 6$
38. $f(x) = x^3 + 3x^2 + 3x - 1$
39. $f(x) = x^3 + 2x^2 + 2x - 1$
40. $f(x) = \dfrac{7x}{x^2 + 4}$
41. $f(x) = x \cdot \sin(x)$ and $0 \le x \le 5$. (You may need to use the Bisection Algorithm or the "trace" option on a calculator to approximate where $f'(x) = 0$.)
42. $f(x) = Ax^2 + Bx + C$, where B and C are constants and $A \neq 0$ is constant.
43. $f(x) = x^3 + Ax^2 + Bx + C$ with constants A, B and C. Can you find conditions on the constants A, B and C that will guarantee that the graph of $y = f(x)$ has two distinct "turning points"? (Here a "turning point" means a place where the curve changes from increasing to decreasing or from decreasing to increasing, like the vertex of a parabola.)

In 44–51, where are the functions differentiable?

44. $f(x) = |x|\cos(x)$
45. $f(x) = \tan(x)$
46. $f(x) = \dfrac{x - 5}{x + 3}$
47. $f(x) = \dfrac{x^2 + x}{x^2 - 3x}$
48. $f(x) = \left|x^2 - 4\right|$
49. $f(x) = \left|x^3 - 1\right|$
50. $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sin(x) & \text{if } x \ge 0 \end{cases}$
51. $f(x) = \begin{cases} x & \text{if } x < 0 \\ \sin(x) & \text{if } x \ge 0 \end{cases}$
52. For what value(s) of A is
$$f(x) = \begin{cases} Ax - 4 & \text{if } x < 2 \\ x^2 + x & \text{if } x \ge 2 \end{cases}$$
differentiable at $x = 2$?
53. For what values of A and B is
$$f(x) = \begin{cases} Ax + B & \text{if } x < 1 \\ x^2 + x & \text{if } x \ge 1 \end{cases}$$
differentiable at $x = 1$?
54. An arrow shot straight up from ground level (get out of the way!) with an initial velocity of 128 feet per second will be at height $h(x) = -16x^2 + 128x$ feet after x seconds (see figure below).
    (a) Determine the velocity of the arrow when $x = 0$, 1 and 2 seconds.
    (b) What is the velocity of the arrow, $v(x)$, at any time x?
    (c) At what time x will the velocity of the arrow be 0?
    (d) What is the greatest height the arrow reaches?
    (e) How long will the arrow be aloft?
    (f) Use the answer for the velocity in part (b) to determine the acceleration, $a(x) = v'(x)$, at any time x.



55. If an arrow is shot straight up from ground level on the moon with an initial velocity of 128 feet per second, its height will be $h(x) = -2.65x^2 + 128x$ feet after x seconds. Redo parts (a)–(e) of problem 40 using this new formula for $h(x)$.

56. In general, if an arrow is shot straight upward with an initial velocity of 128 feet per second from ground level on a planet with a constant gravitational acceleration of g feet per second^2 then its height will be $h(x) = -\frac{g}{2}x^2 + 128x$ feet after x seconds. Answer the questions in problem 40 for arrows shot on Mars and Jupiter.

object	g (ft/sec^2)	g (cm/sec^2)
Mercury	11.8	358
Venus	20.1	887
Earth	32.2	981
moon	5.3	162
Mars	12.3	374
Jupiter	85.3	2601
Saturn	36.6	1117
Uranus	34.4	1049
Neptune	43.5	1325

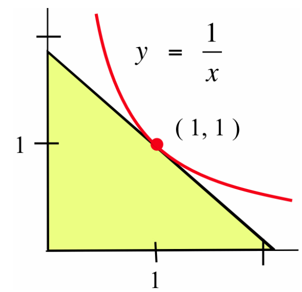
Source: *CRC Handbook of Chemistry and Physics*

57. If an object on Earth is propelled upward from ground level with an initial velocity of v_0 feet per second, then its height after x seconds will be $h(x) = -16x^2 + v_0 x$.
    (a) Find the object's velocity after x seconds.
    (b) Find the greatest height the object will reach.
    (c) How long will the object remain aloft?

58. In order for a 6-foot-tall basketball player to dunk the ball, the player must achieve a vertical jump of about 3 feet. Use the information in the previous problems to answer the following questions.
    (a) What is the smallest initial vertical velocity the player can have and still dunk the ball?
    (b) With the initial velocity achieved in part (a), how high would the player jump on the moon?

59. The best high jumpers in the world manage to lift their centers of mass approximately 3.75 feet.
    (a) What is the initial vertical velocity these high jumpers attain?
    (b) How long are these high jumpers in the air?
    (c) How high would they lift their centers of mass on the moon?

60. (a) Find an equation for the line L that is tangent to the curve $y = \dfrac{1}{x}$ at the point $(1, 1)$.
    (b) Determine where L intersects the x-axis and the y-axis.
    (c) Determine the area of the region in the first quadrant bounded by L, the x-axis and the y-axis (see figure below).



61. (a) Find an equation for the line L that is tangent to the curve $y = \dfrac{1}{x}$ at the point $(2, \frac{1}{2})$.
    (b) Graph $y = \frac{1}{x}$ and L and determine where L intersects the x-axis and the y-axis.
    (c) Determine the area of the region in the first quadrant bounded by L, the x-axis and the y-axis.

62. (a) Find an equation for the line L that is tangent to the curve $y = \dfrac{1}{x}$ at the point $(p, \frac{1}{p})$ (assuming $p \neq 0$).
    (b) Determine where L intersects the x-axis and the y-axis.
    (c) Determine the area of the region in the first quadrant bounded by L, the x-axis and the y-axis.
    (d) How does the area of the triangle in part (c) depend on the initial point $(p, \frac{1}{p})$?

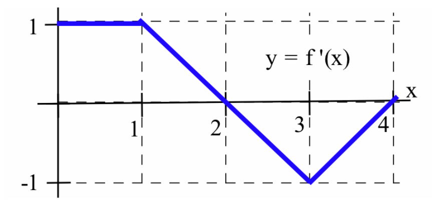
63. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes through the point $(1, 4)$ and is tangent to the line $y = 9x - 13$ at the point $(3, 14)$.

64. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes through the point $(0, 1)$ and is also tangent to the line $y = 3x - 2$ at the point $(2, 4)$.

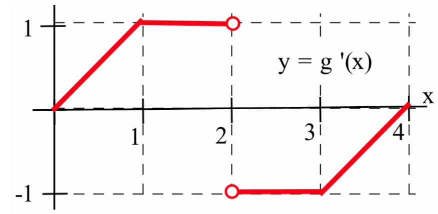
65. (a) Find a function f so that $\mathbf{D}(f(x)) = 3x^2$.
    (b) Find another function g with $\mathbf{D}(g(x)) = 3x^2$.
    (c) Can you find more functions whose derivatives are $3x^2$?

66. (a) Find a function f so that $f'(x) = 6x + \cos(x)$.
    (b) Find another function g with $g'(x) = f'(x)$.

67. The graph of $y = f'(x)$ appears below.
    (a) Assume $f(0) = 0$ and sketch a graph of $y = f(x)$.
    (b) Assume $f(0) = 1$ and graph $y = f(x)$.



68. The graph of $y = g'(x)$ appears below. Assume that g is continuous.
    (a) Assume $g(0) = 0$ and sketch a graph of $y = g(x)$.
    (b) Assume $g(0) = 1$ and graph $y = g(x)$.



69. Assume that f and g are differentiable functions and that $g(x) \neq 0$. State why each step in the following proof of the Quotient Rule is valid.

Proof of the Quotient Rule

$$\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)}\right]$$

$$= \lim_{h\to 0} \frac{1}{g(x+h)g(x)}\left[\frac{f(x+h)g(x) + (-f(x)g(x) + f(x)g(x)) - g(x+h)f(x)}{h}\right]$$

$$= \lim_{h\to 0} \frac{1}{g(x+h)g(x)}\left[g(x)\frac{f(x+h) - f(x)}{h} + f(x)\frac{g(x) - g(x+h)}{h}\right]$$

$$= \frac{1}{[g(x)]^2}\left[g(x)\cdot f'(x) - f(x)\cdot g'(x)\right]$$

$$= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Practice Answers

1. f is continuous at $x = -1, 0, 2, 4, 6$ and 7.
   f is differentiable at $x = -1, 2, 4,$ and 7.

2. $f(x) = 6x + 8$ and $g(x) = 2$ so $\mathbf{D}(f(x)) = 6$ and $\mathbf{D}(g(x)) = 0$.
   $\mathbf{D}(3 \cdot f(x)) = 3 \cdot \mathbf{D}(f(x)) = 3(6) = 18$
   $\mathbf{D}(f(x) + g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x)) = 6 + 0 = 6$
   $\mathbf{D}(f(x) - g(x)) = \mathbf{D}(f(x)) - \mathbf{D}(g(x)) = 6 - 0 = 6$
   $\mathbf{D}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x) = (6x+8)(0) + (2)(6) = 12$
   $\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} = \frac{(2)(6) - (6x+8)(0)}{2^2} = \frac{12}{4} = 3$

3. $\mathbf{D}(x^3 - 5\sin(x)) = \mathbf{D}(x^3) - 5 \cdot \mathbf{D}(\sin(x)) = 3x^2 - 5\cos(x)$
$\frac{d}{dx}\left(\sin(x) - 4x^3\right) = \frac{d}{dx}\sin(x) - 4 \cdot \frac{d}{dx}x^3 = \cos(x) - 12x^2$

4.

$\mathbf{D}(3f(x))$	$\mathbf{D}(2f(x)+g(x))$	$\mathbf{D}(3g(x)-f(x))$
-6	-1	11
-3	-2	1
6	5	1

5.

$\mathbf{D}(f(x) \cdot g(x))$	$\mathbf{D}\left(\frac{f(x)}{g(x)}\right)$	$\mathbf{D}\left(\frac{g(x)}{f(x)}\right)$
$3 \cdot 3 + (-4)(-2) = 17$	$\frac{-4(-2)-(3)(3)}{(-4)^2} = -\frac{1}{16}$	$\frac{(3)(3)-(-4)(-2)}{3^2} = \frac{1}{9}$
$2 \cdot 0 + 1(-1) = -1$	$\frac{1(-1)-(2)(0)}{1^2} = -1$	$\frac{2(0)-1(-1)}{2^2} = \frac{1}{4}$
$4 \cdot 1 + 3 \cdot 2 = 10$	$\frac{3(2)-(4)(1)}{3^2} = \frac{2}{9}$	$\frac{4(1)-3(2)}{4^2} = -\frac{1}{8}$

6. $\mathbf{D}((x^2+1)(7x-3)) = (x^2+1)\,\mathbf{D}(7x-3) + (7x-3)\,\mathbf{D}(x^2+1)$
$= (x^2+1)(7) + (7x-3)(2x) = 21x^2 - 6x + 7$
or: $\mathbf{D}((x^2+1)(7x-3)) = \mathbf{D}(7x^3 - 3x^2 + 7x) = 21x^2 - 6x + 7$
$\frac{d}{dt}\left(\frac{3t-2}{5t+1}\right) = \frac{(5t+1)\,\mathbf{D}(3t-2) - (3t-2)\,\mathbf{D}(5t+1)}{(5t+1)^2} = \frac{(5t+1)(3) - (3t-2)(5)}{(5t+1)^2} = \frac{13}{(5t+1)^2}$
$\mathbf{D}\left(\frac{\cos(x)}{x}\right) = \frac{x\,\mathbf{D}(\cos(x)) - \cos(x)\,\mathbf{D}(x)}{x^2} = \frac{x(-\sin(x)) - \cos(x)(1)}{x^2} = \frac{-x \cdot \sin(x) - \cos(x)}{x^2}$

7. Mimicking the proof of the Difference Rule:

$$\begin{aligned}\mathbf{D}(f(x)+g(x)) &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h} \\ &= \lim_{h\to 0} \frac{[f(x+h)-f(x)] + [g(x+h)-g(x)]}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \mathbf{D}(f(x)) + \mathbf{D}(g(x))\end{aligned}$$

8. (a) difference rule (b) product rule (c) difference rule (d) quotient rule

9. $\mathbf{D}\left(\frac{x^2-5}{\sin(x)}\right) = \frac{\sin(x)\,\mathbf{D}(x^2-5) - (x^2-5)\,\mathbf{D}(\sin(x))}{(\sin(x))^2} = \frac{\sin(x)(2x) - (x^2-5)\cos(x)}{\sin^2(x)}$
$\frac{d}{dt}\left(\frac{t^2-5}{t \cdot \sin(t)}\right) = \frac{t \cdot \sin(t)\,\mathbf{D}(t^2-5) - (t^2-5)\,\mathbf{D}(t \cdot \sin(t))}{(t \cdot \sin(t))^2} = \frac{t \cdot \sin(t)(2t) - (t^2-5)\,[t\cos(t) + \sin(t)]}{t^2 \cdot \sin^2(t)}$

10. $h(5) = 5 + \frac{\sin(5)}{1+5} \approx 4.84$ ft.; $v(5) = h'(5) = \frac{(1+5)\cos(5) - \sin(5)}{(1+5)^2} \approx 0.074$ ft/sec.
"long time": $h(t) = 5 + \frac{\sin(t)}{1+t} \approx 5$ feet when t is very large;
$h'(t) = \frac{(1+t)\cos(t) - \sin(t)}{(1+t)^2} = \frac{\cos(t)}{1+t} - \frac{\sin(t)}{(1+t)^2} \approx 0$ ft/sec when t is very large.

11. $f'(x) = 2x - 10$ so $f'(x) = 0 \Rightarrow 2x - 10 = 0 \Rightarrow x = 5$.
$g'(x) = 3x^2 - 12$ so $g'(x) = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

2.3 *More Differentiation Patterns*

Polynomials are very useful, but they are not the only functions we need. This section uses the ideas of the two previous sections to develop techniques for differentiating **powers of functions**, and to determine the derivatives of some particular functions that occur often in applications: the **trigonometric** and **exponential** functions.

As you focus on learning how to differentiate different types and combinations of functions, it is important to remember what derivatives are and what they measure. Calculators and computers are available to calculate derivatives. Part of your job as a professional will be to decide which functions need to be differentiated and how to use the resulting derivatives. You can succeed at that only if you understand what a derivative is and what it measures.

A Power Rule for Functions: $\mathbf{D}(f^n(x))$

If we apply the Product Rule to the product of a function with itself, a pattern emerges.

$$\begin{array}{lllllll}
\mathbf{D}(f^2) & = & \mathbf{D}(f\cdot f) & = & f\cdot\mathbf{D}(f) + f\cdot\mathbf{D}(f) & = & & = & 2f\cdot\mathbf{D}(f)\\
\mathbf{D}(f^3) & = & \mathbf{D}(f^2\cdot f) & = & f^2\cdot\mathbf{D}(f) + f\cdot\mathbf{D}(f^2) & = & f^2\cdot\mathbf{D}(f) + f\cdot 2f\cdot\mathbf{D}(f) & = & 3f^2\cdot\mathbf{D}(f)\\
\mathbf{D}(f^4) & = & \mathbf{D}(f^3\cdot f) & = & f^3\cdot\mathbf{D}(f) + f\cdot\mathbf{D}(f^3) & = & f^3\cdot\mathbf{D}(f) + f\cdot 3f^2\cdot\mathbf{D}(f) & = & 4f^3\cdot\mathbf{D}(f)
\end{array}$$

Practice 1. What is the pattern here? What do you think the results will be for $\mathbf{D}(f^5)$ and $\mathbf{D}(f^{13})$?

We could keep differentiating higher and higher powers of $f(x)$ by writing them as products of lower powers of $f(x)$ and using the Product Rule, but the Power Rule for Functions guarantees that the pattern we just saw for the small integer powers also works for all constant powers of functions.

The Power Rule for Functions is a special case of a more general theorem, the Chain Rule, which we will examine in Section 2.4, so we will wait until then to prove the Power Rule for Functions.

Power Rule for Functions:

If p is any constant
then $\mathbf{D}(f^p(x)) = p\cdot f^{p-1}(x)\cdot\mathbf{D}(f(x))$.

Example 1. Use the Power Rule for Functions to find:

Remember: $\sin^2(x) = [\sin(x)]^2$

(a) $\mathbf{D}((x^3-5)^2)$ (b) $\dfrac{d}{dx}\left(\sqrt{2x+3x^5}\right)$ (c) $\mathbf{D}(\sin^2(x))$

Solution. (a) To match the pattern of the Power Rule for $\mathbf{D}((x^3-5)^2)$, let $f(x) = x^3-5$ and $p=2$. Then:

Check that you get the same answer by first expanding $(x^3-5)^2$ and then taking the derivative.

$$\begin{aligned}\mathbf{D}((x^3-5)^2) &= \mathbf{D}(f^p(x)) = p\cdot f^{p-1}(x)\cdot\mathbf{D}(f(x))\\ &= 2(x^3-5)^1\,\mathbf{D}(x^3-5) = 2(x^3-5)(3x^2) = 6x^2(x^3-5)\end{aligned}$$

(b) To match the pattern for $\frac{d}{dx}\left(\sqrt{2x+3x^5}\right) = \frac{d}{dx}\left((2x+3x^5)^{\frac{1}{2}}\right)$, let $f(x) = 2x+3x^5$ and take p = $\frac{1}{2}$. Then:

$$\begin{aligned}\frac{d}{dx}\left(\sqrt{2x+3x^5}\right) &= \frac{d}{dx}\left(f^p(x)\right) = p\cdot f^{p-1}(x)\cdot\frac{d}{dx}(f(x))\\ &= \frac{1}{2}(2x+3x^5)^{-\frac{1}{2}}\frac{d}{dx}(2x+3x^5)\\ &= \frac{1}{2}(2x+3x^5)^{-\frac{1}{2}}(2+15x^4) = \frac{2+15x^4}{2\sqrt{2x+3x^5}}\end{aligned}$$

(c) To match the pattern for $\mathbf{D}(\sin^2(x))$, let $f(x) = \sin(x)$ and $p = 2$:

$$\begin{aligned}\mathbf{D}(\sin^2(x)) &= \mathbf{D}(f^p(x)) = p\cdot f^{p-1}(x)\cdot\mathbf{D}(f(x))\\ &= 2\sin^1(x)\,\mathbf{D}(\sin(x)) = 2\sin(x)\cos(x)\end{aligned}$$

We could also rewrite this last expression as $\sin(2x)$. ◀

Practice 2. Use the Power Rule for Functions to find:

(a) $\frac{d}{dx}\left((2x^5-\pi)^2\right)$ (b) $\mathbf{D}\left(\sqrt{x+7x^2}\right)$ (c) $\mathbf{D}(\cos^4(x))$

Example 2. Use calculus to show that the line tangent to the circle $x^2+y^2 = 25$ at the point $(3,4)$ has slope $-\frac{3}{4}$.

Solution. The top half of the circle is the graph of $f(x) = \sqrt{25-x^2}$ so:

$$f'(x) = \mathbf{D}\left((25-x^2)^{\frac{1}{2}}\right) = \frac{1}{2}(25-x^2)^{-\frac{1}{2}}\cdot\mathbf{D}(25-x^2) = \frac{-x}{\sqrt{25-x^2}}$$

and $f'(3) = \frac{-3}{\sqrt{25-3^2}} = -\frac{3}{4}$. As a check, you can verify that the slope of the radial line through the center of the circle $(0,0)$ and the point $(3,4)$ has slope $\frac{4}{3}$ and is perpendicular to the tangent line that has a slope of $-\frac{3}{4}$. ◀

Derivatives of Trigonometric Functions

We have some general rules that apply to any elementary combination of differentiable functions, but in order to use the rules we still need to know the derivatives of some basic functions. Here we will begin to add to the list of functions whose derivatives we know.

We already know the derivatives of the sine and cosine functions, and each of the other four trigonometric functions is just a ratio involving sines or cosines. Using the Quotient Rule, we can easily differentiate the rest of the trigonometric functions.

Theorem:

$$\begin{aligned}\mathbf{D}(\tan(x)) &= \sec^2(x) & \mathbf{D}(\sec(x)) &= \sec(x)\cdot\tan(x)\\ \mathbf{D}(\cot(x)) &= -\csc^2(x) & \mathbf{D}(\csc(x)) &= -\csc(x)\cdot\cot(x)\end{aligned}$$

Proof. From trigonometry, we know $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$, $\sec(x) = \dfrac{1}{\cos(x)}$ and $\csc(x) = \dfrac{1}{\sin(x)}$. From calculus, we already know $\mathbf{D}(\sin(x)) = \cos(x)$ and $\mathbf{D}(\cos(x)) = -\sin(x)$. So:

$$\begin{aligned}\mathbf{D}(\tan(x)) = \mathbf{D}\left(\frac{\sin(x)}{\cos(x)}\right) &= \frac{\cos(x)\cdot\mathbf{D}(\sin(x)) - \sin(x)\cdot\mathbf{D}(\cos(x))}{(\cos(x))^2} \\ &= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)\end{aligned}$$

Similarly:

$$\begin{aligned}\mathbf{D}(\sec(x)) = \mathbf{D}\left(\frac{1}{\cos(x)}\right) &= \frac{\cos(x)\cdot\mathbf{D}(1) - 1\cdot\mathbf{D}(\cos(x))}{(\cos(x))^2} \\ &= \frac{\cos(x)\cdot 0 - (-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)} = \sec(x)\cdot\tan(x)\end{aligned}$$

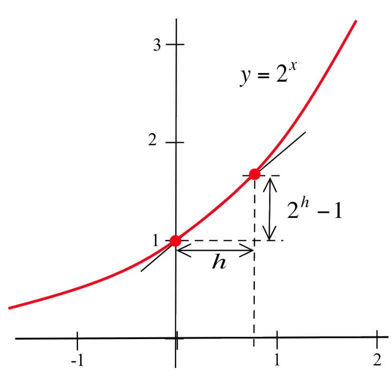
Instead of the Quotient Rule, we could have used the Power Rule to calculate $\mathbf{D}(\sec(x)) = \mathbf{D}((\cos(x))^{-1})$. □

Practice 3. Use the Quotient Rule on $f(x) = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$ to prove that $f'(x) = -\csc^2(x)$.

Practice 4. Prove that $\mathbf{D}(\csc(x)) = -\csc(x)\cdot\cot(x)$. The justification of this result is very similar to the justification for $\mathbf{D}(\sec(x))$.

Practice 5. Find: (a) $\mathbf{D}(x^5\tan(x))$ (b) $\dfrac{d}{dt}\left(\dfrac{\sec(t)}{t}\right)$ (c) $\mathbf{D}\left(\sqrt{\cot(x)-x}\right)$

Derivatives of Exponential Functions



We can estimate the value of a derivative of an exponential function (a function of the form $f(x) = a^x$ where $a > 0$) by estimating the slope of the line tangent to the graph of such a function, or we can numerically approximate those slopes.

Example 3. Estimate the value of the derivative of $f(x) = 2^x$ at the point $(0, 2^0) = (0, 1)$ by approximating the slope of the line tangent to $f(x) = 2^x$ at that point.

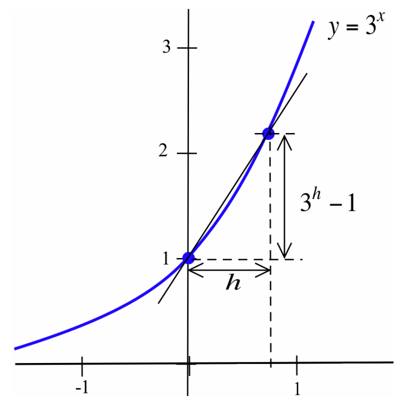
Solution. We can get estimates from the graph of $f(x) = 2^x$ by carefully graphing $y = 2^x$ for small values of x (so that x is near 0), sketching secant lines, and then measuring the slopes of the secant lines (see margin figure).

We can also estimate the slope numerically by using the definition of the derivative:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{2^{0+h} - 2^0}{h} = \lim_{h\to 0} \frac{2^h - 1}{h}$$

and evaluating $\frac{2^h - 1}{h}$ for some very small values of h. From the table below we can see that $f'(0) \approx 0.693$. ◀

h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$	$\frac{e^h-1}{h}$
$+0.1$	0.717734625		
-0.1	0.669670084		
$+0.01$	0.695555006		
-0.01	0.690750451		
$+0.001$	0.693387463		
-0.001	0.692907009		
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	≈ 0.693	≈ 1.099	1



Practice 6. Fill in the table for $\frac{3^h - 1}{h}$ and show that the slope of the line tangent to $g(x) = 3^x$ at $(0, 1)$ is approximately 1.099.

At $(0, 1)$, the slope of the tangent to $y = 2^x$ is less than 1 and the slope of the tangent to $y = 3^x$ is slightly greater than 1. You might expect that there is a number b between 2 and 3 so that the slope of the tangent to $y = b^x$ is exactly 1. Indeed, there is such a number, $e \approx 2.71828182845904$, with

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

The number e is irrational and plays a very important role in calculus and applications.

In fact, e is a "transcendental" number, which means that it is not the root of any polynomial equation with rational coefficients.

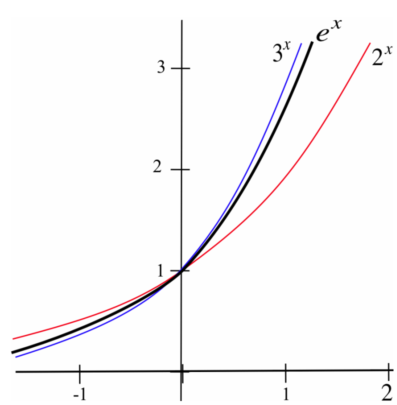
We have not proved that this number e with the desired limit property actually exists, but if we assume it does, then it becomes relatively straightforward to calculate $\mathbf{D}(e^x)$.

Don't worry — we'll tie up some of these loose ends in Chapter 7.

Theorem: $\mathbf{D}(e^x) = e^x$

Proof. Using the definition of the derivative:

$$\mathbf{D}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x \cdot e^h - e^x}{h}$$
$$= \lim_{h\to 0} e^x \cdot \frac{e^h - 1}{h} = \lim_{h\to 0} e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h}$$
$$= e^x \cdot 1 = e^x$$



The function $f(x) = e^x$ is its own derivative: $f'(x) = f(x)$. □

Notice that the limit property of e that we assumed was true actually says that for $f(x) = e^x$, $f'(0) = 1$. So knowing the derivative of $f(x) = e^x$ at a single point ($x = 0$) allows us to determine its derivative at every other point.

Graphically: the **height** of $f(x) = e^x$ at any point and the **slope** of the tangent to $f(x) = e^x$ at that point are the same: as the graph gets higher, its slope gets steeper.

Example 4. Find: (a) $\frac{d}{dt}\left(t \cdot e^t\right)$ (b) $\mathbf{D}\left(\frac{e^x}{\sin(x)}\right)$ (c) $\mathbf{D}(e^{5x})$

Solution. (a) Using the Product Rule with $f(t) = t$ and $g(t) = e^t$:

$$\frac{d}{dt}\left(t \cdot e^t\right) = t \cdot \mathbf{D}(e^t) + e^t \cdot \mathbf{D}(t) = t \cdot e^t + e^t \cdot 1 = (t+1)e^t$$

(b) Using the Quotient Rule with $f(x) = e^x$ and $g(x) = \sin(x)$:

$$\mathbf{D}\left(\frac{e^x}{\sin(x)}\right) = \frac{\sin(x) \cdot \mathbf{D}(e^x) - e^x \cdot \mathbf{D}(\sin(x))}{[\sin(x)]^2} = \frac{\sin(x) \cdot e^x - e^x(\cos(x))}{\sin^2(x)}$$

(c) Using the Power Rule for Functions with $f(x) = e^x$ and $p = 5$:

$$\mathbf{D}((e^x)^5) = 5(e^x)^4 \cdot \mathbf{D}(e^x) = 5e^{4x} \cdot e^x = 5e^{5x}$$

where we have rewritten e^{5x} as $(e^x)^5$. ◀

Practice 7. Find: (a) $\mathbf{D}(x^3 e^x)$ (b) $\mathbf{D}((e^x)^3)$.

Higher Derivatives: Derivatives of Derivatives

The derivative of a function f is a new function f' and if this new function is differentiable we can calculate the derivative of this new function to get the derivative of the derivative of f, denoted by f'' and called the **second derivative** of f.

For example, if $f(x) = x^5$ then $f'(x) = 5x^4$ and:

$$f''(x) = (f'(x))' = (5x^4)' = 20x^3$$

Definitions: Given a differentiable function f,

- the first derivative is $f'(x)$, the rate of change of f.
- the second derivative is $f''(x) = (f'(x))'$, the rate of change of f'.
- the third derivative is $f'''(x) = (f''(x))'$, the rate of change of f''.

For $y = f(x)$, we write $f'(x) = \frac{dy}{dx}$, so we can extend that notation to write $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$, $f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ and so on.

Practice 8. Find f', f'' and f''' for $f(x) = 3x^7$, $f(x) = \sin(x)$ and $f(x) = x \cdot \cos(x)$.

If $f(x)$ represents the position of a particle at time x, then $v(x) = f'(x)$ will represent the velocity (rate of change of the position) of the particle and $a(x) = v'(x) = f''(x)$ will represent the acceleration (the rate of change of the velocity) of the particle.

Example 5. The height (in feet) of a particle at time t seconds is given by $t^3 - 4t^2 + 8t$. Find the height, velocity and acceleration of the particle when $t = 0$, 1 and 2 seconds.

Solution. $f(t) = t^3 - 4t^2 + 8t$ so $f(0) = 0$ feet, $f(1) = 5$ feet and $f(2) = 8$ feet. The velocity is given by $v(t) = f'(t) = 3t^2 - 8t + 8$ so $v(0) = 8$ ft/sec, $v(1) = 3$ ft/sec and $v(2) = 4$ ft/sec. At each of these times the velocity is positive and the particle is moving upward (increasing in height). The acceleration is $a(t) = 6t - 8$ so $a(0) = -8$ ft/sec^2, $a(1) = -2$ ft/sec^2 and $a(2) = 4$ ft/sec^2. ◀

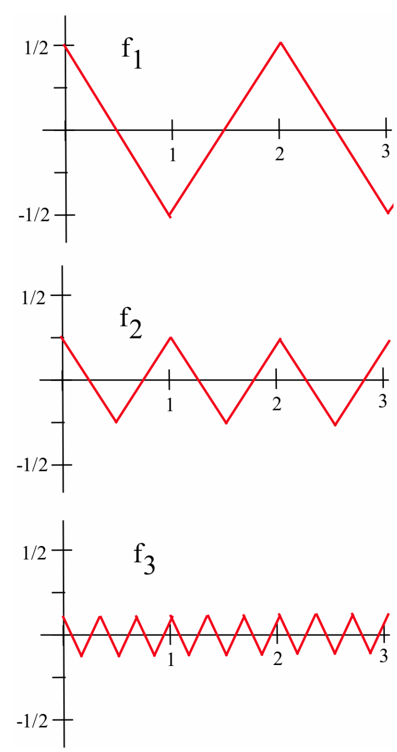
We will examine the geometric (graphical) meaning of the second derivative in the next chapter.

A Really "Bent" Function

In Section 1.2 we saw that the "holey" function

$$h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

is discontinuous at every value of x, so $h(x)$ is not differentiable anywhere. We can create graphs of continuous functions that are not differentiable at several places just by putting corners at those places, but how many corners can a continuous function have? How badly can a continuous function fail to be differentiable?



In the mid-1800s, the German mathematician Karl Weierstrass surprised and even shocked the mathematical world by creating a function that was **continuous everywhere but differentiable nowhere**—a function whose graph was everywhere connected and everywhere bent! He used techniques we have not investigated yet, but we can begin to see how such a function could be built.

Start with a function f_1 (see margin) that zigzags between the values $\frac{1}{2}$ and $-\frac{1}{2}$ and has a "corner" at each integer. This starting function f_1 is continuous everywhere and is differentiable everywhere except at the integers. Next create a list of functions $f_2, f_3, f_4, \ldots,$ each of which is "shorter" than the previous one but with many more "corners" than the previous one. For example, we might make f_2 zigzag between the values $\frac{1}{4}$ and $-\frac{1}{4}$ and have "corners" at $\pm\frac{1}{2}$, $\pm\frac{3}{2}$, $\pm\frac{5}{2}$, etc.; f_3 zigzag between $\frac{1}{9}$ and $-\frac{1}{9}$ and have "corners" at $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{3}{3} = \pm 1$, etc.

If we add f_1 and f_2, we get a continuous function (because the sum of two continuous functions is continuous) with corners at 0, $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$, ... If we then add f_3 to the previous sum, we get a new continuous function with even more corners. If we continue adding the functions in our list "indefinitely," the final result will be a continuous function that is differentiable nowhere.

We haven't developed enough mathematics here to precisely describe what it means to add an infinite number of functions together or to verify that the resulting function is nowhere differentiable—but we will. You can at least start to imagine what a strange, totally "bent" function it must be. Until Weierstrass created his "everywhere continuous, nowhere differentiable" function, most mathematicians thought a continuous function could only be "bad" in a few places. Weierstrass' function was (and is) considered "pathological," a great example of how bad something can be. The mathematician Charles Hermite expressed a reaction shared by many when they first encounter the Weierstrass function: "I turn away with fright and horror from this lamentable evil of functions which do not have derivatives."

Important Results

Power Rule for Functions: $\mathbf{D}(f^p(x)) = p \cdot f^{p-1}(x) \cdot \mathbf{D}(f(x))$

Derivatives of the Trigonometric Functions:

$$\begin{array}{ll} \mathbf{D}(\sin(x)) = \cos(x) & \mathbf{D}(\cos(x)) = -\sin(x) \\ \mathbf{D}(\tan(x)) = \sec^2(x) & \mathbf{D}(\cot(x)) = -\csc^2(x) \\ \mathbf{D}(\sec(x)) = \sec(x)\tan(x) & \mathbf{D}(\csc(x)) = -\csc(x)\cot(x) \end{array}$$

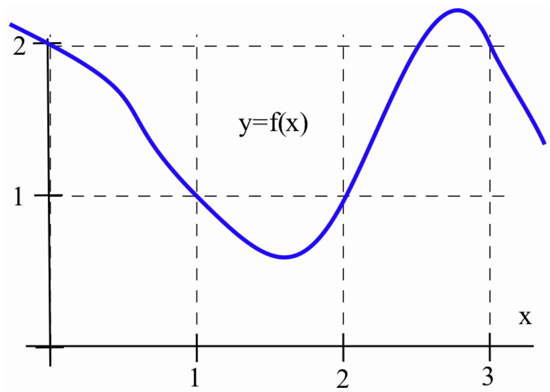
Derivative of the Exponential Function: $\mathbf{D}(e^x) = e^x$

2.3 Problems

1. Let $f(1) = 2$ and $f'(1) = 3$. Find the values of each of the following derivatives at $x = 1$.
   (a) $\mathbf{D}(f^2(x))$
   (b) $\mathbf{D}(f^5(x))$
   (c) $\mathbf{D}(\sqrt{f(x)})$

2. Let $f(2) = -2$ and $f'(2) = 5$. Find the values of each of the following derivatives at $x = 2$.
   (a) $\mathbf{D}(f^2(x))$
   (b) $\mathbf{D}(f^{-3}(x))$
   (c) $\mathbf{D}(\sqrt{f(x)})$

3. For $x = 1$ and $x = 3$ estimate the values of $f(x)$ (whose graph appears below), $f'(x)$ and
(a) $\frac{d}{dx}\left(f^2(x)\right)$ (b) $\mathbf{D}\left(f^3(x)\right)$ (c) $\mathbf{D}\left(f^5(x)\right)$



4. For $x = 0$ and $x = 2$ estimate the values of $f(x)$ (whose graph appears above), $f'(x)$ and
(a) $\mathbf{D}\left(f^2(x)\right)$ (b) $\frac{d}{dx}\left(f^3(x)\right)$ (c) $\frac{d}{dx}\left(f^5(x)\right)$

In Problems 5–10, find $f'(x)$.

5. $f(x) = (2x - 8)^5$
6. $f(x) = (6x - x^2)^{10}$
7. $f(x) = x \cdot (3x + 7)^5$
8. $f(x) = (2x + 3)^6 \cdot (x - 2)^4$
9. $f(x) = \sqrt{x^2 + 6x - 1}$
10. $f(x) = \dfrac{x - 5}{(x + 3)^4}$
11. A mass attached to the end of a spring is at a height of $h(t) = 3 - 2\sin(t)$ feet above the floor t seconds after it is released.
    - (a) Graph $h(t)$.
    - (b) At what height is the mass when it is released?
    - (c) How high does above the floor and how close to the floor does the mass ever get?
    - (d) Determine the height, velocity and acceleration at time t. (Be sure to include the correct units.)
    - (e) Why is this an unrealistic model of the motion of a mass attached to a real spring?
12. A mass attached to a spring is at a height of $h(t) = 3 - \dfrac{2\sin(t)}{1 + 0.1t^2}$ feet above the floor t seconds after it is released.
    - (a) Graph $h(t)$.
    - (b) At what height is the mass when it is released?
    - (c) Determine the velocity of the mass at time t.
    - (d) What happens to the height and the velocity of the mass a "long time" after it is released?
13. The kinetic energy K of an object of mass m and velocity v is $\frac{1}{2}mv^2$.
    - (a) Find the kinetic energy of an object with mass m and height $h(t) = 5t$ feet at $t = 1$ and $t = 2$ seconds.
    - (b) Find the kinetic energy of an object with mass m and height $h(t) = t^2$ feet at $t = 1$ and $t = 2$ seconds.
14. An object of mass m is attached to a spring and has height $h(t) = 3 + \sin(t)$ feet at time t seconds.
    - (a) Find the height and kinetic energy of the object when $t = 1$, 2 and 3 seconds.
    - (b) Find the rate of change in the kinetic energy of the object when $t = 1$, 2 and 3 seconds.
    - (c) Can K ever be negative? Can $\dfrac{dK}{dt}$ ever be negative? Why?

In Problems 15–20, compute $f'(x)$.

15. $f(x) = x \cdot \sin(x)$
16. $f(x) = \sin^5(x)$
17. $f(x) = e^x - \sec(x)$
18. $f(x) = \sqrt{\cos(x) + 1}$
19. $f(x) = e^{-x} + \sin(x)$
20. $f(x) = \sqrt{x^2 - 4x + 3}$

In Problems 21–26, find an equation for the line tangent to the graph of $y = f(x)$ at the given point.

21. $f(x) = (x - 5)^7$ at $(4, -1)$
22. $f(x) = e^x$ at $(0, 1)$

23. $f(x) = \sqrt{25 - x^2}$ at $(3, 4)$
24. $f(x) = \sin^3(x)$ at $(\pi, 0)$
25. $f(x) = (x - a)^5$ at $(a, 0)$
26. $f(x) = x \cdot \cos^5(x)$ at $(0, 0)$
27. (a) Find an equation for the line tangent to $f(x) = e^x$ at the point $(3, e^3)$.
    (b) Where will this tangent line intersect the x-axis?
    (c) Where will the tangent line to $f(x) = e^x$ at the point (p, e^p) intersect the x-axis?

In Problems 28–33, calculate f' and f''.

28. $f(x) = 7x^2 + 5x - 3$
29. $f(x) = \cos(x)$
30. $f(x) = \sin(x)$
31. $f(x) = x^2 \cdot \sin(x)$
32. $f(x) = x \cdot \sin(x)$
33. $f(x) = e^x \cdot \cos(x)$
34. Calculate the first 8 derivatives of $f(x) = \sin(x)$. What is the pattern? What is the 208th derivative of $\sin(x)$?
35. What will the second derivative of a quadratic polynomial be? The third derivative? The fourth derivative?
36. What will the third derivative of a cubic polynomial be? The fourth derivative?
37. What can you say about the n-th and $(n+1)$-st derivatives of a polynomial of degree n?

In Problems 38–42, you are given f'. Find a function f with the given derivative.

38. $f'(x) = 4x + 2$
39. $f'(x) = 5e^x$
40. $f'(x) = 3 \cdot \sin^2(x) \cdot \cos(x)$
41. $f'(x) = 5(1 + e^x)^4 \cdot e^x$
42. $f'(x) = e^x + \sin(x)$
43. The function $f(x)$ defined as

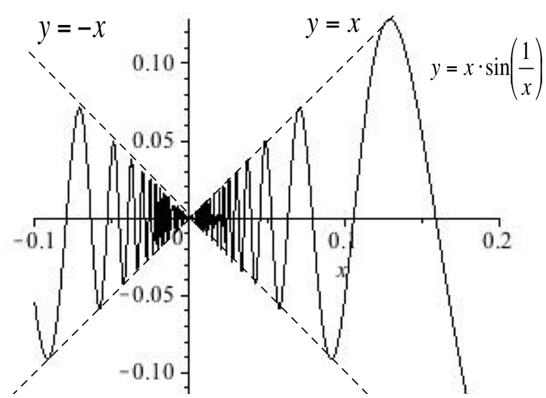
$$f(x) = \begin{cases} x \cdot \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

shown below is continuous at 0 because we can show (using the Squeezing Theorem) that

$$\lim_{h \to 0} f(x) = 0 = f(0)$$

Is f differentiable at 0? To answer this question, use the definition of $f'(0)$ and consider

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$



44. The function $f(x)$ defined as

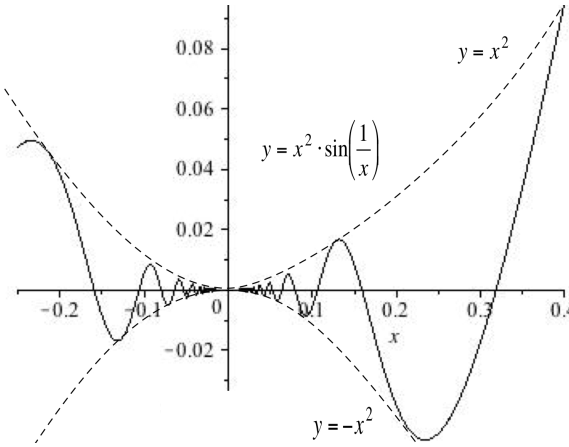
$$f(x) = \begin{cases} x^2 \cdot \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(shown at the top of the next page) is continuous at 0 because we can show (using the Squeezing Theorem) that

$$\lim_{h \to 0} f(x) = 0 = f(0)$$

Is f differentiable at 0? To answer this question, use the definition of $f'(0)$ and consider

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$



The number e appears in a variety of unusual situations. Problems 45–48 illustrate a few of these.

45. Use your calculator to examine the values of $f(x) = \left(1 + \frac{1}{x}\right)^x$ when x is relatively large (for example, $x = 100$, 1000 and 10,000. Try some other large values for x. If x is large, the value of $f(x)$ is close to what number?

46. If you put $1 into a bank account that pays 1% interest per year and compounds the interest x times a year, then after one year you will have $\left(1 + \frac{0.01}{x}\right)^x$ dollars in the account.
   (a) How much money will you have after one year if the bank calculates the interest once a year?
   (b) How much money will you have after one year if the bank calculates the interest twice a year?
   (c) How much money will you have after one year if the bank calculates the interest 365 times a year?
   (d) How does your answer to part (c) compare with $e^{0.01}$?

47. Define $n!$ to be the product of all positive integers from 1 through n. For example, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$ and $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.
   (a) Calculate the value of the sums:

$$s_1 = 1 + \frac{1}{1!}$$
$$s_2 = 1 + \frac{1}{1!} + \frac{1}{2!}$$
$$s_3 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$$
$$s_4 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$$
$$s_5 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$$
$$s_6 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}$$

   (b) What value do the sums in part (a) seem to be approaching?
   (c) Calculate s_7 and s_8.

48. If it is late at night and you are tired of studying calculus, try the following experiment with a friend. Take the 2 through 10 of hearts from a regular deck of cards and shuffle these nine cards well. Have your friend do the same with the 2 through 10 of spades. Now compare your cards one at a time. If there is a match, for example you both play a 5, then the game is over and you win. If you make it through the entire nine cards with no match, then your friend wins. If you play the game **many times**, then the ratio:

$$\frac{\text{total number of games played}}{\text{number of times your friend wins}}$$

will be approximately equal to e.

2.3 Practice Answers

1. The pattern is $\mathbf{D}(f^n(x)) = n \cdot f^{n-1}(x) \cdot \mathbf{D}(f(x))$:
$\mathbf{D}(f^5(x)) = 5f^4(x) \cdot \mathbf{D}(f(x))$ and $\mathbf{D}(f^{13}(x)) = 13f^{12}(x) \cdot \mathbf{D}(f(x))$

2. $\frac{d}{dx}(2x^5 - \pi)^2 = 2(2x^5 - \pi)^1\, \mathbf{D}(2x^5 - \pi) = 2(2x^5 - \pi)(10x^4) = 40x^9 - 20\pi x^4$
$\mathbf{D}\left((x + 7x^2)^{\frac{1}{2}}\right) = \frac{1}{2}(x + 7x^2)^{-\frac{1}{2}}\, \mathbf{D}(x + 7x^2) = \frac{1 + 14x}{2\sqrt{x + 7x^2}}$
$\mathbf{D}\left((\cos(x))^4\right) = 4(\cos(x))^3\, \mathbf{D}(\cos(x)) = 4(\cos(x))^3(-\sin(x)) = -4\cos^3(x)\sin(x)$

3. Mimicking the proof for the derivative of $\tan(x)$:

$$\mathbf{D}\left(\frac{\cos(x)}{\sin(x)}\right) = \frac{\sin(x)\cdot\mathbf{D}(\cos(x)) - \cos(x)\cdot\mathbf{D}(\sin(x))}{(\sin(x))^2}$$
$$= \frac{\sin(x)(-\sin(x)) - \cos(x)(\cos(x))}{\sin^2(x)}$$
$$= \frac{-(\sin^2(x)+\cos^2(x))}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

4. Mimicking the proof for the derivative of $\sec(x)$:

$$\mathbf{D}(\csc(x)) = \mathbf{D}\left(\frac{1}{\sin(x)}\right) = \frac{\sin(x)\cdot\mathbf{D}(1) - 1\cdot\mathbf{D}(\sin(x)}{\sin^2(x)}$$
$$= \frac{\sin(x)\cdot 0 - \cos(x)}{\sin^2(x)} = -\frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)} = -\cot(x)\csc(x)$$

5. $\mathbf{D}(x^5\cdot\tan(x)) = x^5\,\mathbf{D}(\tan(x)) + \tan(x)\,\mathbf{D}(x^5) = x^5\sec^2(x) + \tan(x)(5x^4)$

$$\frac{d}{dt}\left(\frac{\sec(t)}{t}\right) = \frac{t\,\mathbf{D}(\sec(t)) - \sec(t)\,\mathbf{D}(t)}{t^2} = \frac{t\sec(t)\tan(t) - \sec(t)}{t^2}$$

$$\mathbf{D}\left((\cot(x)-x)^{\frac{1}{2}}\right) = \frac{1}{2}(\cot(x)-x)^{-\frac{1}{2}}\,\mathbf{D}(\cot(x)-x)$$
$$= \frac{1}{2}(\cot(x)-x)^{-\frac{1}{2}}(-\csc^2(x)-1) = \frac{-\csc^2(x)-1}{2\sqrt{\cot(x)-x}}$$

6. Filling in values for both 3^x and e^x:

h	$\frac{2^h-1}{h}$	$\frac{3^h-1}{h}$	$\frac{e^h-1}{h}$
$+0.1$	0.717734625	1.161231740	1.0517091808
-0.1	0.669670084	1.040415402	0.9516258196
$+0.01$	0.695555006	1.104669194	1.0050167084
-0.01	0.690750451	1.092599583	0.9950166251
$+0.001$	0.693387463	1.099215984	1.0005001667
-0.001	0.692907009	1.098009035	0.9995001666
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	≈ 0.693	≈ 1.0986	1

7. $\mathbf{D}(x^3e^x) = x^3\,\mathbf{D}(e^x) + e^x\,\mathbf{D}(x^3) = x^3e^x + e^x\cdot 3x^2 = x^2e^x(x+3)$
$\mathbf{D}\left((e^x)^3\right) = 3\,(e^x)^2\,\mathbf{D}(e^x) = 3e^{2x}\cdot e^x = 3e^{3x}$

8. $f(x) = 3x^7 \Rightarrow f'(x) = 21x^6 \Rightarrow f''(x) = 126x^5 \Rightarrow f'''(x) = 630x^4$
$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x)$
$\Rightarrow f'''(x) = -\cos(x)$
$f(x) = x\cdot\cos(x) \Rightarrow f'(x) = -x\sin(x) + \cos(x)$
$\Rightarrow f''(x) = -x\cos(x) - 2\sin(x) \Rightarrow f'''(x) = x\sin(x) - 3\cos(x)$

2.4 *The Chain Rule*

The Chain Rule is the **most important and most often used** of the differentiation patterns. It enables us to differentiate **composites** of functions such as $y = \sin(x^2)$. It is a powerful tool for determining the derivatives of some **new functions** such as logarithms and inverse trigonometric functions. And it leads to important **applications** in a variety of fields. You will need the Chain Rule hundreds of times in this course. Practice with it now will save you time—and points—later. Fortunately, with practice, the Chain Rule is also easy to use. We already know how to differentiate the composition of some functions.

Example 1. For $f(x) = 5x - 4$ and $g(x) = 2x + 1$, find $f \circ g(x)$ and $\mathbf{D}(f \circ g(x))$.

Solution. Writing $f \circ g(x) = f(g(x)) = 5(2x+1) - 4 = 10x + 1$, we can compute that $\mathbf{D}(f \circ g(x)) = \mathbf{D}(10x+1) = 10$. ◀

Practice 1. For $f(x) = 5x - 4$ and $g(x) = x^2$, find $f \circ g(x)$, $\mathbf{D}(f \circ g(x))$, $g \circ f(x)$ and $\mathbf{D}(g \circ f(x))$.

Some compositions, however, are still very difficult to differentiate. We know the derivatives of $g(x) = x^2$ and $h(x) = \sin(x)$, and we know how to differentiate certain combinations of these functions, such as $x^2 + \sin(x)$, $x^2 \cdot \sin(x)$ and even $\sin^2(x) = (\sin(x))^2$. But the derivative of the simple composition $f(x) = h \circ g(x) = \sin(x^2)$ is hard—until we know the Chain Rule.

To see just how difficult, try using the definition of derivative on it.

Example 2. (a) Suppose amplifier Y doubles the strength of the output signal from amplifier U, and U triples the strength of the original signal x. How does the final signal out of Y compare with the original signal x?

(b) Suppose y changes twice as fast as u, and u changes three times as fast as x. How does the rate of change of y compare with the rate of change of x?

Solution. In each case we are comparing the result of a composition, and the answer to each question is 6, the product of the two amplifications or rates of change. In part (a), we have that:

$$\frac{\text{signal out of } Y}{\text{signal } x} = \frac{\text{signal out of } Y}{\text{signal out of } U} \cdot \frac{\text{signal out of } U}{\text{signal } x} = 2 \cdot 3 = 6$$

In part (b):

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = 2 \cdot 3 = 6$$

These examples are simple cases of the Chain Rule for differentiating a composition of functions. ◀

The Chain Rule

We can express the chain rule using more than one type of notation. Each will be useful in various situations.

> **Chain Rule (Leibniz notation form)**:
>
> If y is a differentiable function of u and
> u is a differentiable function of x
> then y is a differentiable function of x and
> $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Idea for a proof. If $\Delta u \neq 0$ then:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} = \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \right) \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right)$$
$$= \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \right) \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right) = \frac{dy}{du} \cdot \frac{du}{dx}$$

The key step here is to argue that $\Delta x \to 0$ implies $\Delta u \to 0$, which follows from the continuity of u as as function of x.

Although this nice short argument gets to the heart of why the Chain Rule works, it is not quite valid. If $\frac{du}{dx} \neq 0$, then it is possible to show that $\Delta u \neq 0$ for all "very small" values of Δx, and the "idea for a proof" becomes a real proof. There are, however, functions for which $\Delta u = 0$ for infinitely many small values of Δx (no matter how close to 0 we restrict Δx) and this creates problems with the simple argument outlined above.

A justification that holds true for **all** cases is more complicated and provides no new conceptual insight. Problem 84 at the end of this section guides you through a rigorous proof of the Chain Rule.

The symbol $\frac{dy}{du}$ is a single symbol, as is $\frac{du}{dx}$, so we cannot eliminate du from the product $\frac{dy}{du}\frac{du}{dx}$ in the Chain Rule by "cancelling" du as we can with Δu in the fractions $\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$. It is, however, perfectly fine to use the *idea* of cancelling du to help you remember the proper statement of the Chain Rule.

Example 3. Write $y = \cos(x^2 + 3)$ as $y = \cos(u)$ with $u = x^2 + 3$ and find $\frac{dy}{dx}$.

Solution. $y = \cos(u) \Rightarrow \frac{dy}{du} = -\sin(u)$ and $u = x^2 + 3 \Rightarrow \frac{du}{dx} = 2x$. Using the Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u) \cdot 2x = -2x \cdot \sin(x^2 + 3)$$

Notice that in the last step we have eliminated the intermediate variable u to express the derivative only in terms of x. ◀

Practice 2. Find $\frac{dy}{dx}$ for $y = \sin(4x + e^x)$.

We can also state the Chain Rule in terms of composition of functions. The notation is different, but the meaning is precisely the same.

Chain Rule (composition form):

If	g is differentiable at x and
	f is differentiable at $g(x)$
then	the composite $f \circ g$ is differentiable at x and
	$(f \circ g)'(x) = \mathbf{D}(f(g(x))) = f'(g(x)) \cdot g'(x)$.

You may find it easier to think of the result of the composition form of the Chain Rule in words: "the derivative of the outside function (evaluated at the original inside function) times the derivative of the inside function" where f is the outside function and g is the inside function.

Example 4. Differentiate $\sin(x^2)$.

Solution. We can write the function $\sin(x^2)$ as the composition $f \circ g$ of two simple functions: $f(x) = \sin(x)$ and $g(x) = x^2$: $f \circ g(x) = f(g(x)) = f(x^2) = \sin(x^2)$. Both f and g are differentiable functions with derivatives $f'(x) = \cos(x)$ and $g'(x) = 2x$, so the Chain Rule says:

$$\mathbf{D}(\sin(x^2)) = (f \circ g)'(x) = f'(g(x)) \cdot g'(x) = \cos(g(x)) \cdot 2x$$
$$= \cos(x^2) \cdot 2x = 2x\cos(x^2)$$

Check that you get the same answer using the Leibniz notation. ◀

If you tried using the definition of derivative to calculate the derivative of this function at the beginning of this section, you can really appreciate the power of the Chain Rule for differentiating compositions of functions, even simple ones like these.

Example 5. The table below gives values for f, f', g and g' at various points. Use these values to determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -1$ and $x = 0$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
-1	2	3	1	0		
0	-1	1	3	2		
1	1	0	-1	3		
2	3	-1	0	1		
3	0	2	2	-1		

Solution. $(f \circ g)(-1) = f(g(-1)) = f(3) = 0$, $(f \circ g)(0) = f(g(0)) = f(1) = 1$, $(f \circ g)'(-1) = f'(g(-1)) \cdot g'(-1) = f'(3) \cdot 0 = 2 \cdot 0 = 0$ and $(f \circ g)'(0) = f'(g(0)) \cdot g'(0) = f'(1) \cdot 2 = (-1)(2) = -2$. ◀

Practice 3. Fill in the table in Example 5 for $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1, 2$ and 3.

Neither form of the Chain Rule is inherently superior to the other — use the one you prefer or the one that appears most useful in a particular situation. The Chain Rule will be used hundreds of times in the rest of this book, and it is important that you master its usage. The time you spend now mastering and understanding how to use the Chain Rule will be paid back tenfold over the next several chapters.

Example 6. Determine $\mathbf{D}\left(e^{\cos(x)}\right)$ using each form of the Chain Rule.

Solution. Using the Leibniz notation: $y = e^u$ and $u = \cos(x)$ so we have $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = -\sin(x)$. Applying the Chain Rule:

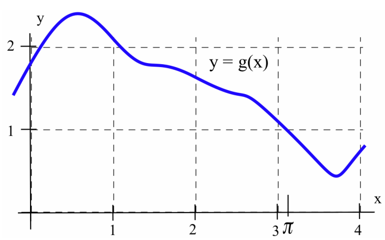
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (-\sin(x)) = -\sin(x) \cdot e^{\cos(x)}$$

We can also write the function $e^{\cos(x)}$ as the composition of $f(x) = e^x$ with $g(x) = \cos(x)$, so the Chain Rule says:

$$\mathbf{D}(e^{\cos(x)}) = f'(g(x)) \cdot g'(x) = e^{g(x)} \cdot (-\sin(x)) = -\sin(x) \cdot e^{\cos(x)}$$

because $\mathbf{D}(e^x) = e^x$ and $\mathbf{D}(\cos(x)) = -\sin(x)$. ◀

Practice 4. Calculate $\mathbf{D}(\sin(7x-1))$, $\frac{d}{dx}(\sin(ax+b))$ and $\frac{d}{dt}\left(e^{3t}\right)$.



Practice 5. Use the graph of g given in the margin along with the Chain Rule to estimate $\mathbf{D}(\sin(g(x)))$ and $\mathbf{D}(g(\sin(x)))$ at $x = \pi$.

The Chain Rule is a general differentiation pattern that can be used along with other general patterns like the Product and Quotient Rules.

Example 7. Determine $\mathbf{D}\left(e^{3x} \cdot \sin(5x+7)\right)$ and $\frac{d}{dx}(\cos(x \cdot e^x))$.

Solution. The function $e^{3x}\sin(5x+7)$ is a product of two functions so we need the Product Rule first:

$$\begin{aligned}\mathbf{D}(e^{3x} \cdot \sin(5x+7)) &= e^{3x} \cdot \mathbf{D}(\sin(5x+7)) + \sin(5x+7) \cdot \mathbf{D}(e^{3x}) \\ &= e^{3x} \cdot \cos(5x+7) \cdot 5 + \sin(5x+7) \cdot e^{3x} \cdot 3 \\ &= 5e^{3x}\cos(5x+7) + 3e^{3x}\sin(5x+7)\end{aligned}$$

The function $\cos(x \cdot e^x)$ is a composition of cosine with a product so we need the Chain Rule first:

$$\begin{aligned}\frac{d}{dx}(\cos(x \cdot e^x)) &= -\sin(x \cdot e^x) \cdot \frac{d}{dx}(x \cdot e^x) \\ &= -\sin(xe^x) \cdot \left(x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x)\right) \\ &= -\sin(xe^x) \cdot (xe^x + e^x)\end{aligned}$$

We could also write this last answer as $-(x+1)e^x\sin(e^x)$. ◀

Sometimes we want to differentiate a composition of more than two functions. We can do so if we proceed in a careful, step-by-step way.

Example 8. Find $\mathbf{D}(\sin(\sqrt{x^3+1}))$.

Solution. The function $\sin(\sqrt{x^3+1})$ can be viewed as a composition $f \circ g$ of $f(x) = \sin(x)$ and $g(x) = \sqrt{x^3+1}$. Then:

$$\begin{aligned}(\sin(\sqrt{x^3+1}))' &= f'(g(x)) \cdot g'(x) = \cos(g(x)) \cdot g'(x) \\ &= \cos(\sqrt{x^3+1}) \cdot \mathbf{D}(\sqrt{x^3+1})\end{aligned}$$

For the derivative of $\sqrt{x^3+1}$, we can use the Chain Rule again or its special case, the Power Rule:

$$\begin{aligned}\mathbf{D}(\sqrt{x^3+1}) = \mathbf{D}((x^3+1)^{\frac{1}{2}}) &= \frac{1}{2}(x^3+1)^{-\frac{1}{2}} \cdot \mathbf{D}(x^3+1) \\ &= \frac{1}{2}(x^3+1)^{-\frac{1}{2}} \cdot 3x^2\end{aligned}$$

Finally, $\mathbf{D}\left(\sin(\sqrt{x^3+1})\right) = \cos(\sqrt{x^3+1}) \cdot \frac{1}{2}(x^3+1)^{-\frac{1}{2}} \cdot 3x^2$, which can be rewritten as $\dfrac{3x^2\cos(\sqrt{x^3+1})}{2\sqrt{x^3+1}}$. ◀

This example was more complicated than the earlier ones, but it is just a matter of applying the Chain Rule twice, to a composition of a composition. If you proceed step by step and don't get lost in the details of the problem, these multiple applications of the Chain Rule are relatively straightforward.

We can also use the Leibniz form of the Chain Rule for a composition of more than two functions. If $y = \sin(\sqrt{x^3+1})$, then $y = \sin(u)$ with $u = \sqrt{w}$ and $w = x^3 + 1$. The Leibniz form of the Chain Rule says:

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx} &= \cos(u) \cdot \frac{1}{2\sqrt{w}} \cdot 3x^2 \\ &= \cos(\sqrt{x^3+1}) \cdot \frac{1}{2\sqrt{x^3+1}} \cdot 3x^2\end{aligned}$$

which agrees with our previous answer.

Practice 6. (a) Find $\mathbf{D}(\sin(\cos(5x)))$. (b) For $y = e^{\cos(3x)}$, find $\frac{dy}{dx}$.

The Chain Rule and Tables of Derivatives

With the Chain Rule, the derivatives of all sorts of strange and wonderful functions become available. If we know f' and g', then we also know the derivatives of their composition: $(f(g(x))' = f'(g(x)) \cdot g'(x)$.

We have begun to build a list of derivatives of "basic" functions, such as x^n, $\sin(x)$ and e^x. We will continue to add to that list later in the course, but if we peek ahead at the rest of that list—spoiler alert!—to (for example) see that $\mathbf{D}(\arctan(x)) = \frac{1}{1+x^2}$, then we can use the Chain Rule to compute derivatives of compositions of those functions.

Example 9. Given that $\mathbf{D}(\arcsin(x)) = \dfrac{1}{\sqrt{1-x^2}}$, compute the derivatives $\mathbf{D}(\arcsin(5x))$ and $\dfrac{d}{dx}(\arcsin(e^x))$.

Solution. Write $\arcsin(5x)$ as the composition of $f(x) = \arcsin(x)$ with $g(x) = 5x$. We know $g'(x) = 5$ and $f'(x) = \dfrac{1}{\sqrt{1-x^2}}$, so we have $f'(g(x)) = \dfrac{1}{\sqrt{1-(g(x))^2}} = \dfrac{1}{\sqrt{1-25x^2}}$. Then:

$$\mathbf{D}(\arcsin(5x)) = f'(g(x)) \cdot g'(x) = \frac{1}{\sqrt{1-(5x)^2}} \cdot 5 = \frac{5}{\sqrt{1-25x^2}}$$

We can write $y = \arcsin(e^x)$ as $y = \arcsin(u)$ with $u = e^x$, and we know that $\dfrac{dy}{du} = \dfrac{1}{\sqrt{1-u^2}}$ and $\dfrac{du}{dx} = e^x$ so:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \cdot e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$$

We can generalize this result to say that $\mathbf{D}(\arcsin(f(x))) = \dfrac{f'(x)}{\sqrt{1-(f(x))^2}}$ or, in Leibniz notation, $\dfrac{d}{du}(\arcsin(u)) = \dfrac{1}{\sqrt{1-u^2}} \cdot \dfrac{du}{dx}$. ◀

Practice 7. Given that $\mathbf{D}(\arctan(x)) = \dfrac{1}{1+x^2}$, compute the derivatives $\mathbf{D}(\arctan(x^3))$ and $\dfrac{d}{dx}(\arctan(e^x))$.

Appendix D in the back of this book shows the derivative patterns for a variety of functions. You may not know much about some of the functions, but with the given differentiation patterns and the Chain Rule you should be able to calculate derivatives of compositions that involve these new functions. It is just a matter of following the pattern.

Practice 8. Use the patterns $\mathbf{D}(\sinh(x)) = \cosh(x)$ and $\mathbf{D}(\ln(x)) = \frac{1}{x}$ to determine:

(a) $\mathbf{D}(\sinh(5x-7))$ (b) $\dfrac{d}{dx}\left(\ln(3+e^{2x})\right)$ (c) $\mathbf{D}(\arcsin(1+3x))$

Example 10. If $\mathbf{D}(F(x)) = e^x \cdot \sin(x)$, find $\mathbf{D}(F(5x))$ and $\dfrac{d}{dt}\left(F(t^3)\right)$.

Solution. $\mathbf{D}(F(5x)) = \mathbf{D}(F(g(x))$ with $g(x) = 5x$ and we know that $F'(x) = e^x \cdot \sin(x)$ so:

$$\mathbf{D}(F(5x)) = F'(g(x)) \cdot g'(x) = e^{g(x)} \cdot \sin(g(x)) \cdot 5 = e^{5x} \cdot \sin(5x) \cdot 5$$

With $y = F(u)$ and $u = t^3$ we know $\dfrac{dy}{du} = e^u \cdot \sin(u)$ and $\dfrac{du}{dt} = 3t^2$ so:

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = e^u \cdot \sin(u) \cdot 3t^2 = e^{t^3} \cdot \sin(t^3) \cdot 3t^2$$

Notice that we have eliminated the intermediate variable u (which didn't appear in the original problem) from the final answer. ◀

Proof of the Power Rule For Functions

We started using the Power Rule For Functions in Section 2.3. Now we can easily prove it.

> **Power Rule For Functions**:
>
> If $\quad p$ is any constant
>
> then $\quad \mathbf{D}(f^p(x)) = p \cdot f^{p-1}(x) \cdot \mathbf{D}(f(x))$.

Proof. Write $y = f^p(x)$ as $y = u^p$ with $u = f(x)$. Then $\dfrac{dy}{du} = p \cdot u^{p-1}$ and $\dfrac{du}{dx} = f'(x)$ so:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = p \cdot u^{p-1} \cdot f'(x) = p \cdot f^{p-1}(x) \cdot f'(x)$$

by the Chain Rule. □

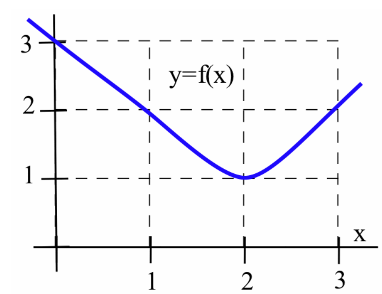
2.4 *Problems*

In Problems 1–6 , find two functions f and g so that the given function is the composition of f and g.

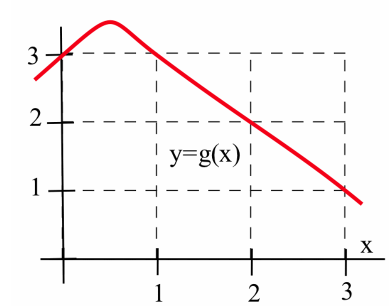
1. $y = (x^3 - 7x)^5$
2. $y = \sin^4(3x - 8)$
3. $y = \sqrt{(2 + \sin(x))^5}$
4. $y = \dfrac{1}{\sqrt{x^2 + 9}}$
5. $y = |x^2 - 4|$
6. $y = \tan(\sqrt{x})$
7. For each function in Problems 1–6, write y as a function of u for some u that is a function of x.

For Problems 8–9, use the values given in this table to determine the indicated quantities:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
−2	2	−1	1	1		
−1	1	2	0	2		
0	−2	1	2	−1		
1	0	−2	−1	2		
2	1	0	1	−1		



8. $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1$ and $x = 2$.
9. $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -2$, -1 and 0.



10. Using the figure in the margin, estimate the values of $g(x)$, $g'(x)$, $(f \circ g)(x)$, $f'(g(x))$ and $(f \circ g)'(x)$ at $x = 1$.
11. Using the figure in the margin, estimate the values of $g(x)$, $g'(x)$, $(f \circ g)(x)$, $f'(g(x))$ and $(f \circ g)'(x)$ at $x = 2$.

In Problems 12–22, compute the derivative.

12. $\mathbf{D}\left((x^2+2x+3)^{87}\right)$

13. $\mathbf{D}\left(\left(1-\frac{3}{x}\right)^4\right)$

14. $\frac{d}{dx}\left(x+\frac{1}{x}\right)^5$

15. $\mathbf{D}\left(\frac{5}{\sqrt{2+\sin(x)}}\right)$

16. $\frac{d}{dt}\left(t\cdot\sin(3t+2)\right)$

17. $\frac{d}{dx}\left(x^2\cdot\sin(x^2+3)\right)$

18. $\frac{d}{dx}\left(\sin(2x)\cdot\cos(5x+1)\right)$

19. $\mathbf{D}\left(\frac{7}{\cos(x^3-x)}\right)$

20. $\frac{d}{dt}\left(\frac{5}{3+e^t}\right)$

21. $\mathbf{D}\left(e^x+e^{-x}\right)$

22. $\mathbf{D}\left(e^x-e^{-x}\right)$

23. An object attached to a spring is at a height of $h(t)=3-\cos(2t)$ feet above the floor t seconds after it is released.
   (a) At what height was it released?
   (b) Determine its height, velocity and acceleration at any time t.
   (c) If the object has mass m, determine its kinetic energy $K=\frac{1}{2}mv^2$ and $\frac{dK}{dt}$ at any time t.

24. An employee with d days of production experience will be able to produce approximately $P(d)=3+15(1-e^{-0.2d})$ items per day.
   (a) Graph $P(d)$.
   (b) Approximately how many items will a beginning employee be able to produce each day?
   (c) How many items will a very experienced employee be able to produce each day?
   (d) What is the marginal production rate of an employee with 5 days of experience? (Include units for your answer. What does this mean?)

25. The air pressure $P(h)$, in pounds per square inch, at an altitude of h feet above sea level is approximately $P(h)=14.7e^{-0.0000385h}$.
   (a) What is the air pressure at sea level?
   (b) What is the air pressure at 30,000 feet?
   (c) At what altitude is the air pressure 10 pounds per square inch?
   (d) If you are in a balloon that is 2,000 feet above the Pacific Ocean and is rising at 500 feet per minute, how fast is the air pressure on the balloon changing?
   (e) If the temperature of the gas in the balloon remained constant during this ascent, what would happen to the volume of the balloon?

Find the indicated derivatives in Problems 26–33.

26. $\mathbf{D}\left(\frac{(2x+3)^2}{(5x-7)^3}\right)$

27. $\frac{d}{dz}\sqrt{1+\cos^2(z)}$

28. $\mathbf{D}\left(\sin(3x+5)\right)$

29. $\frac{d}{dx}\tan(3x+5)$

30. $\frac{d}{dt}\cos(7t^2)$

31. $\mathbf{D}\left(\sin(\sqrt{x+1})\right)$

32. $\mathbf{D}\left(\sec(\sqrt{x+1})\right)$

33. $\frac{d}{dx}\left(e^{\sin(x)}\right)$

In Problems 34–37 , calculate $f'(x)\cdot x'(t)$ when $t=3$ and use these values to determine the value of $\frac{d}{dt}\left(f(x(t))\right)$ when $t=3$.

34. $f(x)=\cos(x)$, $x=t^2-t+5$
35. $f(x)=\sqrt{x}$, $x=2+\frac{21}{t}$
36. $f(x)=e^x$, $x=\sin(t)$
37. $f(x)=\tan^3(x)$, $x=8$

In 38–43, find a function that has the given function as its derivative. (You are given a function $f'(x)$ and are asked to find a corresponding function $f(x)$.)

38. $f'(x)=(3x+1)^4$

39. $f'(x)=(7x-13)^{10}$

40. $f'(x)=\sqrt{3x-4}$

41. $f'(x)=\sin(2x-3)$

42. $f'(x)=6e^{3x}$

43. $f'(x)=\cos(x)e^{\sin(x)}$

If two functions are equal, then their derivatives are also equal. In 44–47 , differentiate each side of the trigonometric identity to get a new identity.

44. $\sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x)$
45. $\cos(2x)=\cos^2(x)-\sin^2(x)$
46. $\sin(2x)=2\sin(x)\cdot\cos(x)$
47. $\sin(3x)=3\sin(x)-4\sin^3(x)$

Derivatives of Families of Functions

So far we have emphasized derivatives of particular functions, but sometimes we want to investigate the derivatives of a whole family of functions all at once. In 48–71, A, B, C and D represent constants and the given formulas describe families of functions.

For Problems 48–65, calculate $y' = \frac{dy}{dx}$.

48. $y = Ax^3 - B$
49. $y = Ax^3 + Bx^2 + C$
50. $y = \sin(Ax + B)$
51. $y = \sin(Ax^2 + B)$
52. $y = Ax^3 + \cos(Bx)$
53. $y = \sqrt{A + Bx^2}$
54. $y = \sqrt{A - Bx^2}$
55. $y = A - \cos(Bx)$
56. $y = \cos(Ax + B)$
57. $y = \cos(Ax^2 + B)$
58. $y = A \cdot e^{Bx}$
59. $y = x \cdot e^{Bx}$
60. $y = e^{Ax} + e^{-Ax}$
61. $y = e^{Ax} - e^{-Ax}$
62. $y = \dfrac{\sin(Ax)}{x}$
63. $y = \dfrac{Ax}{\sin(Bx)}$
64. $y = \dfrac{1}{Ax + B}$
65. $y = \dfrac{Ax + B}{Cx + D}$

In 66–71, (a) find y' (b) find the value(s) of x so that $y' = 0$ and (c) find y''. Typically your answer in part (b) will contain A's, B's and (sometimes) C's.

66. $y = Ax^2 + Bx + C$
67. $y = Ax(B - x) = ABx - Ax^2$
68. $y = Ax(B - x^2) = ABx - Ax^3$
69. $y = Ax^2(B - x) = ABx^2 - Ax^3$
70. $y = Ax^2 + Bx$
71. $y = Ax^3 + Bx^2 + C$

In Problems 72–83, use the differentiation patterns $\mathbf{D}(\arctan(x)) = \dfrac{1}{1+x^2}$, $\mathbf{D}(\arcsin(x)) = \dfrac{1}{\sqrt{1-x^2}}$ and $\mathbf{D}(\ln(x)) = \dfrac{1}{x}$. We have not derived the derivatives for these functions (yet), but if you are handed the derivative pattern then you should be able to use that pattern to compute derivatives of associated composite functions.

72. $\mathbf{D}\left(\arctan(7x)\right)$
73. $\mathbf{D}\left(\arctan(x^2)\right)$
74. $\dfrac{d}{dt}\left(\arctan(\ln(t))\right)$
75. $\dfrac{d}{dx}\left(\arctan(e^x)\right)$
76. $\dfrac{d}{dw}\left(\arcsin(4w)\right)$
77. $\dfrac{d}{dx}\left(\arcsin(x^3)\right)$
78. $\mathbf{D}\left(\arcsin(\ln(x))\right)$
79. $\mathbf{D}\left(\arcsin(e^t)\right)$
80. $\mathbf{D}\left(\ln(3x+1)\right)$
81. $\mathbf{D}\left(\ln(\sin(x))\right)$
82. $\dfrac{d}{dx}\left(\ln(\arctan(x))\right)$
83. $\dfrac{d}{ds}\left(\ln(e^s)\right)$
84. To prove the Chain Rule, assume $g(x)$ is differentiable at $x = a$ and $f(x)$ is differentiable at $x = g(a)$. We need to show that
$$\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a}$$
exists and is equal to $f'(g(a)) \cdot g'(a)$. To do this, define a new function F as:
$$F(y) = \begin{cases} \frac{f(y) - f(g(a))}{y - g(a)} & \text{if } y \neq g(a) \\ f'(g(a)) & \text{if } y = g(a) \end{cases}$$
and justify each of the following statements.
(a) $F(y)$ is continuous at $y = g(a)$ because:
$$\lim_{y\to g(a)} F(y) = \lim_{y\to g(a)} \frac{f(y) - f(g(a))}{y - g(a)} = F(g(a))$$
(b) By considering separately the cases $g(x) = g(a)$ and $g(x) \neq g(a)$:
$$\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x)) \cdot \frac{g(x) - g(a)}{x - a}$$
for all $x \neq a$.
(c) $\displaystyle\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x\to a} F(g(x)) \cdot \frac{g(x) - g(a)}{x - a}$
(d) $\displaystyle\lim_{x\to a} F(g(x)) \cdot \frac{g(x) - g(a)}{x - a} = F(g(a)) \cdot g'(a)$
(e) $\displaystyle\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a} = f'(g(a)) \cdot g'(a)$
(f) $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$

2.4 Practice Answers

1. $f(x) = 5x - 4$ and $g(x) = x^2 \Rightarrow f'(x) = 5$ and $g'(x) = 2x$, so $f \circ g(x) = f(g(x)) = f(x^2)) = 5x^2 - 4$ and $\mathbf{D}(5x^2 - 4) = 10x$ or:
$$\mathbf{D}(f \circ g(x)) = f'(g(x)) \cdot g'(x) = 5 \cdot 2x = 10x$$
$g \circ f(x) = g(f(x)) = g(5x-4) = (5x-4)^2 = 25x^2 - 40x + 16$ and $\mathbf{D}(25x^2 - 40x + 16) = 50x - 40$ or:
$$\mathbf{D}(g \circ f(x)) = g'(f(x)) \cdot f'(x) = 2(5x-4) \cdot 5 = 50x - 40$$

2. $\frac{d}{dx}\left(\sin(4x + e^x)\right) = \cos(4x + e^x) \cdot \mathbf{D}(4x + e^x) = \cos(4x + e^x) \cdot (4 + e^x)$

3. To fill in the last column, compute:
$$f'(g(1)) \cdot g'(1) = f'(0) \cdot 3 = (3)(3) = 9$$
$$f'(g(2)) \cdot g'(2) = f'(-1) \cdot 1 = (1)(1) = 1$$
$$f'(g(3) \cdot g'(3) = f'(2) \cdot (-1) = (0)(-1) = 0$$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$	$(f \circ g)(x)$	$(f \circ g)'(x)$
1	1	0	−1	3	−1	9
2	3	−1	0	1	2	3
3	0	2	2	−1	3	0

4. $\mathbf{D}\left(\sin(7x-1)\right) = \cos(7x-1) \cdot \mathbf{D}(7x-1) = 7 \cdot \cos(7x-1)$
$\frac{d}{dx}\left(\sin(ax+b)\right) = \cos(ax+b) \cdot \mathbf{D}(ax+b) = a \cdot \cos(ax+b)$
$\frac{d}{dt}\left(e^{3t}\right) = e^{3t} \cdot \frac{d}{dt}(3t) = 3 \cdot e^{3t}$

5. $\mathbf{D}\left(\sin(g(x))\right) = \cos(g(x)) \cdot g'(x)$. At $x = \pi$, $\cos(g(\pi)) \cdot g'(\pi) \approx \cos(0.86) \cdot (-1) \approx -0.65$. $\mathbf{D}\left(g(\sin(x))\right) = g'(\sin(x)) \cdot \cos(x)$. At $x = \pi$, $g'(\sin(\pi)) \cdot \cos(\pi) = g'(0) \cdot (-1) \approx -2$

6. $\mathbf{D}\left(\sin(\cos(5x))\right) = \cos(\cos(5x)) \cdot \mathbf{D}(\cos(5x))$
$= \cos(\cos(5x)) \cdot (-\sin(5x)) \cdot \mathbf{D}(5x) = -5 \cdot \sin(5x) \cdot \cos(\cos(5x))$
$\frac{d}{dx}\left(e^{\cos(3x)}\right) = e^{\cos(3x)} \cdot \mathbf{D}(\cos(3x)) = e^{\cos(3x)}(-\sin(3x))\,\mathbf{D}(3x)$
$= -3 \cdot \sin(3x) \cdot e^{\cos(3x)}$

7. $\mathbf{D}\left(\arctan(x^3)\right) = \frac{1}{1+(x^3)^2} \cdot \mathbf{D}(x^3) = \frac{3x^2}{1+x^6}$
$\frac{d}{dx}\left(\arctan(e^x)\right) = \frac{1}{1+(e^x)^2} \cdot \mathbf{D}(e^x) = \frac{e^x}{1+e^{2x}}$

8. $\mathbf{D}(\sinh(5x-7)) = \cosh(5x-7) \cdot \mathbf{D}(5x-7) = 5 \cdot \cosh(5x-7)$
$\frac{d}{dx}\left(\ln(3+e^{2x}\right) = \frac{1}{3+e^{2x}} \cdot \mathbf{D}(3+e^{2x}) = \frac{2e^{2x}}{3+e^{2x}}$
$\mathbf{D}(\arcsin(1+3x)) = \frac{1}{\sqrt{1-(1+3x)^2}} \cdot \mathbf{D}(1+3x) = \frac{3}{\sqrt{1-(1+3x)^2}}$

2.5 *Applications of the Chain Rule*

The Chain Rule can help us determine the derivatives of logarithmic functions like $f(x) = \ln(x)$ and general exponential functions like $f(x) = a^x$. We will also use it to answer some applied questions and to find slopes of graphs given by parametric equations.

Derivatives of Logarithms

You know from precalculus that the natural logarithm $\ln(x)$ is defined as the inverse of the exponential function e^x: $e^{\ln(x)} = x$ for $x > 0$. We can use this identity along with the Chain Rule to determine the derivative of the natural logarithm.

$$\mathbf{D}(\ln(x)) = \frac{1}{x} \quad \text{and} \quad \mathbf{D}\left(\ln(g(x))\right) = \frac{g'(x)}{g(x)}$$

Proof. We know that $\mathbf{D}(e^u) = e^u$, so using the Chain Rule we have $\mathbf{D}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$. Differentiating each side of the identity $e^{\ln(x)} = x$, we get:

$$\mathbf{D}\left(e^{\ln(x)}\right) = \mathbf{D}(x) \Rightarrow e^{\ln(x)} \cdot \mathbf{D}(\ln(x)) = 1$$
$$\Rightarrow x \cdot \mathbf{D}(\ln(x)) = 1 \Rightarrow \mathbf{D}(\ln(x)) = \frac{1}{x}$$

The function $\ln(g(x))$ is the composition of $f(x) = \ln(x)$ with $g(x)$ so the Chain Rule says:

$$\mathbf{D}\left(\ln(g(x)\right) = \mathbf{D}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

You can remember the differentiation pattern for the the natural logarithm in words as: "one over the inside times the the derivative of the inside."

Graph $f(x) = \ln(x)$ along with $f'(x) = \frac{1}{x}$ and compare the behavior of the function at various points with the values of its derivative at those points. Does $y = \frac{1}{x}$ possess the properties you would expect to see from the derivative of $f(x) = \ln(x)$? □

Example 1. Find $\mathbf{D}(\ln(\sin(x)))$ and $\mathbf{D}(\ln(x^2 + 3))$.

Solution. Using the pattern $\mathbf{D}(\ln(g(x)) = \frac{g'(x)}{g(x)}$ with $g(x) = \sin(x)$:

$$\mathbf{D}(\ln(\sin(x))) = \frac{g'(x)}{g(x)} = \frac{\mathbf{D}(\sin(x))}{\sin(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

With $g(x) = x^2 + 3$, $\mathbf{D}(\ln(x^2 + 3)) = \frac{g'(x)}{g(x)} = \frac{2x}{x^2 + 3}$. ◀

We can use the Change of Base Formula from precalculus to rewrite any logarithm as a natural logarithm, and then we can differentiate the resulting natural logarithm.

Change of Base Formula for Logarithms:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \text{ for all positive } a, b \text{ and } x.$$

Your calculator likely has two logarithm buttons: **ln** for the natural logarithm (base e) and **log** for the common logarithm (base 10). Be careful, however, as more advanced mathematics texts (as well as the Web site Wolfram | Alpha) use log for the (base e) natural logarithm.

Example 2. Use the Change of Base formula and your calculator to find $\log_\pi(7)$ and $\log_2(8)$.

Solution. $\log_\pi(7) = \dfrac{\ln(7)}{\ln(\pi)} \approx \dfrac{1.946}{1.145} \approx 1.700$. (Check that $\pi^{1.7} \approx 7$.)
Likewise, $\log_2(8) = \dfrac{\ln(8)}{\ln(2)} = 3$. ◀

Practice 1. Find the values of $\log_9 20$, $\log_3 20$ and $\log_\pi e$.

Putting $b = e$ in the Change of Base Formula, $\log_a(x) = \dfrac{\log_e(x)}{\log_e(a)} = \dfrac{\ln(x)}{\ln(a)}$, so any logarithm can be written as a natural logarithm divided by a constant. This makes any logarithmic function easy to differentiate.

$$\mathbf{D}\left(\log_a(x)\right) = \frac{1}{x\ln(a)} \quad \text{and} \quad \mathbf{D}\left(\log_a(f(x))\right) = \frac{f'(x)}{f(x)} \cdot \frac{1}{\ln(a)}$$

Proof. $\mathbf{D}\left(\log_a(x)\right) = \mathbf{D}\left(\dfrac{\ln x}{\ln a}\right) = \dfrac{1}{\ln(a)} \cdot \mathbf{D}(\ln x) = \dfrac{1}{\ln(a)} \cdot \dfrac{1}{x} = \dfrac{1}{x\ln(a)}$.
The second differentiation formula follows from the Chain Rule. □

Practice 2. Calculate $\mathbf{D}\left(\log_{10}(\sin(x))\right)$ and $\mathbf{D}\left(\log_\pi(e^x)\right)$.

The number e might seem like an "unnatural" base for a natural logarithm, but of all the possible bases, the logarithm with base e has the nicest and easiest derivative. The natural logarithm is even related to the distribution of prime numbers. In 1896, the mathematicians Hadamard and Vallée-Poussin proved the following conjecture of Gauss (the Prime Number Theorem): For large values of N,

$$\text{number of primes less than } N \approx \frac{N}{\ln(N)}$$

Derivative of a^x

Once we know the derivative of e^x and the Chain Rule, it is relatively easy to determine the derivative of a^x for any $a > 0$.

$$\mathbf{D}(a^x) = a^x \cdot \ln(a) \text{ for } a > 0.$$

Proof. If $a > 0$, then $a^x > 0$ and $a^x = e^{\ln(a^x)} = e^{x \cdot \ln(a)}$, so we have: $\mathbf{D}(a^x) = \mathbf{D}\left(e^{\ln(a^x)}\right) = \mathbf{D}\left(e^{x \cdot \ln(a)}\right) = e^{x \cdot \ln(a)} \cdot \mathbf{D}(x \cdot \ln(a)) = a^x \cdot \ln(a)$. □

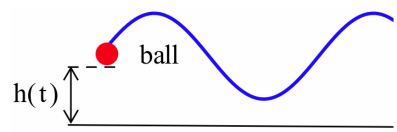
Example 3. Calculate $\mathbf{D}(7^x)$ and $\frac{d}{dt}\left(2^{\sin(t)}\right)$.

Solution. $\mathbf{D}(7^x) = 7^x \cdot \ln(7) \approx (1.95)7^x$. We can write $y = 2^{\sin(t)}$ as $y = 2^u$ with $u = \sin(t)$. Using the Chain Rule: $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = 2^u \cdot \ln(2)\cos(t) = 2^{\sin(t)} \cdot \ln(2) \cdot \cos(t)$. ◄

Practice 3. Calculate $\mathbf{D}\left(\sin(2^x)\right)$ and $\frac{d}{dt}\left(3^{t^2}\right)$.

Some Applied Problems

Let's examine some applications involving more complicated functions.



Example 4. A ball at the end of a rubber band (see margin) is oscillating up and down, and its height (in feet) above the floor at time t seconds is $h(t) = 5 + 2\sin\left(\frac{t}{2}\right)$ (with t in radians).

(a) How fast is the ball traveling after 2 seconds? After 4 seconds? After 60 seconds?

(b) Is the ball moving up or down after 2 seconds? After 4 seconds? After 60 seconds?

(c) Is the vertical velocity of the ball ever 0?

Solution. (a) $v(t) = h'(t) = \mathbf{D}\left(5 + 2\sin\left(\frac{t}{2}\right)\right) = 2\cos\left(\frac{t}{2}\right) \cdot \frac{1}{2}$ so $v(t) = \cos\left(\frac{t}{2}\right)$ feet/second: $v(2) = \cos(\frac{2}{2}) \approx 0.540$ ft/s, $v(4) = \cos(\frac{4}{2}) \approx -0.416$ ft/s, and $v(60) = \cos(\frac{60}{2}) \approx 0.154$ ft/s.

(b) The ball is moving up at $t = 2$ and $t = 60$, down when $t = 4$.

(c) $v(t) = \cos\left(\frac{t}{2}\right) = 0$ when $\frac{t}{2} = \frac{\pi}{2} \pm k \cdot \pi \Rightarrow t = \pi \pm 2\pi k$ for any integer k. ◄

Example 5. If 2,400 people now have a disease, and the number of people with the disease appears to double every 3 years, then the number of people expected to have the disease in t years is $y = 2400 \cdot 2^{\frac{t}{3}}$.

(a) How many people are expected to have the disease in 2 years?

(b) When are 50,000 people expected to have the disease?

(c) How fast is the number of people with the disease growing now? How fast is it expected to be growing 2 years from now?

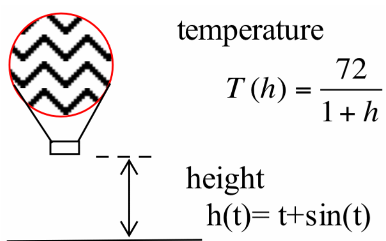
Solution. (a) In 2 years, $y = 2400 \cdot 2^{\frac{2}{3}} \approx$ 3,810 people.

(b) We know $y = 50000$ and need to solve $50000 = 2400 \cdot 2^{\frac{t}{3}}$ for t. Taking logarithms of each side of the equation: $\ln(50000) = \ln\left(2400 \cdot 2^{\frac{2}{3}}\right) = \ln(2400) + \frac{t}{3} \cdot \ln(2)$ so $10.819 \approx 7.783 + 0.231t$ and $t \approx 13.14$ years. We expect 50,000 people to have the disease about 13 years from now.

(c) This question asks for $\frac{dy}{dt}$ when $t = 0$ and $t = 2$.

$$\frac{dy}{dt} = \frac{d}{dt}\left(2400 \cdot 2^{\frac{t}{3}}\right) = 2400 \cdot 2^{\frac{t}{3}} \cdot \ln(2) \cdot \frac{1}{3} \approx 554.5 \cdot 2^{\frac{t}{3}}$$

Now, at $t = 0$, the rate of growth of the disease is approximately $554.5 \cdot 2^0 \approx 554.5$ people/year. In 2 years, the rate of growth will be approximately $554.5 \cdot 2^{\frac{2}{3}} \approx 880$ people/year. ◀



Example 6. You are riding in a balloon, and at time t (in minutes) you are $h(t) = t + \sin(t)$ thousand feet above sea level. If the temperature at an elevation h is $T(h) = \frac{72}{1+h}$ degrees Fahrenheit, then how fast is the temperature changing when $t = 5$ minutes?

Solution. As t changes, your elevation will change. And, as your elevation changes, so will the temperature. It is not difficult to write the temperature as a function of time, and then we could calculate $\frac{dT}{dt} = T'(t)$ and evaluate $T'(5)$. Or we could use the Chain Rule:

$$\frac{dT}{dt} = \frac{dT}{dh} \cdot \frac{dh}{dt} = -\frac{72}{(1+h)^2} \cdot (1 + \cos(t))$$

At $t = 5$, $h(5) = 5 + \sin(5) \approx 4.04$ so $T'(5) \approx -\frac{72}{(1+4.04)^2} \cdot (1 + 0.284) \approx -3.64$ °/minute. ◀

Practice 4. Write the temperature T in the previous example as a function of the variable t alone and then differentiate T to determine the value of $\frac{dT}{dt}$ when $t = 5$ minutes.

Example 7. A scientist has determined that, under optimum conditions, an initial population of 40 bacteria will grow "exponentially" to $f(t) = 40 \cdot e^{\frac{t}{5}}$ bacteria after t hours.

(a) Graph $y = f(t)$ for $0 \le t \le 15$. Calculate $f(0)$, $f(5)$ and $f(10)$.

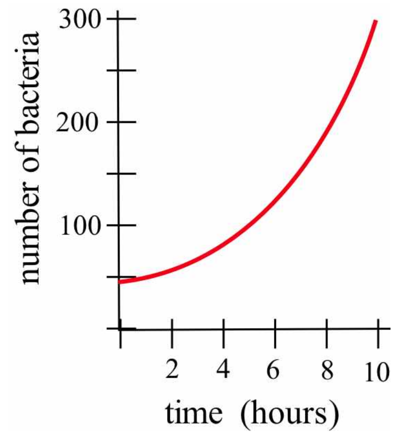
(b) How fast is the population increasing at time t? (Find $f'(t)$.)

(c) Show that the rate of population increase, $f'(t)$, is proportional to the population, $f(t)$, at any time t. (Show $f'(t) = K \cdot f(t)$ for some constant K.)

Solution. (a) The graph of $y = f(t)$ appears in the margin. $f(0) = 40 \cdot e^{\frac{0}{5}} = 40$ bacteria, $f(5) = 40 \cdot e^{\frac{5}{5}} = 40e \approx 109$ bacteria and $f(10) = 40 \cdot e^{\frac{10}{5}} \approx 296$ bacteria.

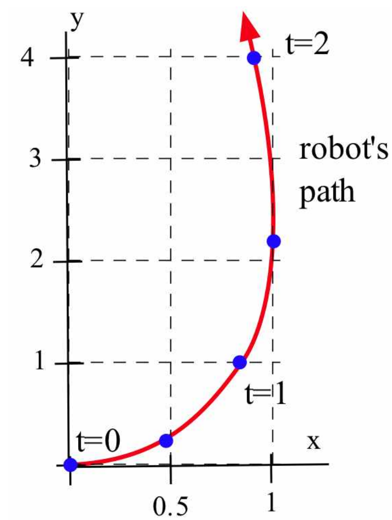
(b) $f'(t) = \frac{d}{dt}(f(t)) = \frac{d}{dt}\left(40 \cdot e^{\frac{t}{5}}\right) = 40 \cdot e^{\frac{t}{5}} \cdot \frac{d}{dt}\left(\frac{t}{5}\right) = 40 \cdot e^{\frac{t}{5}} \cdot \frac{1}{5} = 8 \cdot e^{\frac{t}{5}}$ bacteria/hour.

(c) $f'(t) = 8 \cdot e^{\frac{t}{5}} = \frac{1}{5} \cdot 40e^{\frac{t}{5}} = \frac{1}{5}f(t)$ so $f'(t) = K \cdot f(t)$ with $K = \frac{1}{5}$. The rate of change of the population is proportional to its size. ◀



Parametric Equations

Suppose a robot has been programmed to move in the xy-plane so at time t its x-coordinate will be $\sin(t)$ and its y-coordinate will be t^2. Both x and y are functions of the independent parameter t: $x(t) = \sin(t)$ and $y(t) = t^2$. The path of the robot (see margin) can be found by plotting $(x, y) = (x(t), y(t))$ for lots of values of t.



t	$x(t) = \sin(t)$	$y(t) = t^2$	point
0	0	0	$(0, 0)$
0.5	0.48	0.25	$(0.48, 0.25)$
1.0	0.84	1	$(0.84, 1)$
1.5	1.00	2.25	$(1, 2.25)$
2.0	0.91	4	$(0.91, 4)$

Typically we know $x(t)$ and $y(t)$ and need to find $\frac{dy}{dx}$, the slope of the tangent line to the graph of $(x(t), y(t))$. The Chain Rule says:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

so , algebraically solving for $\frac{dy}{dx}$, we get:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

If we can calculate $\frac{dy}{dt}$ and $\frac{dx}{dt}$, the derivatives of y and x with respect to the parameter t, then we can determine $\frac{dy}{dx}$, the rate of change of y with respect to x.

> If $x = x(t)$ and $y = y(t)$ are differentiable with respect to t and $\frac{dx}{dt} \neq 0$
>
> then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Example 8. Find the slope of the tangent line to the graph of $(x, y) = (\sin(t), t^2)$ when $t = 2$.

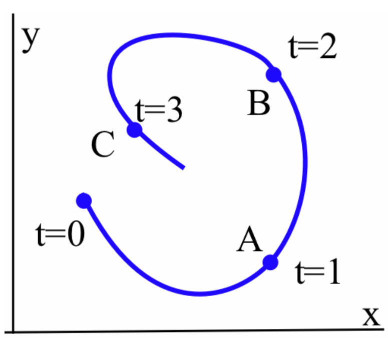
Solution. $\frac{dx}{dt} = \cos(t)$ and $\frac{dy}{dt} = 2t$. When $t = 2$, the object is at the point $(\sin(2), 2^2) \approx (0.91, 4)$ and the slope of the tangent line is:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\cos(t)} = \frac{2 \cdot 2}{\cos(2)} \approx \frac{4}{-0.42} \approx -9.61$$

Notice in the figure that the slope of the tangent line to the curve at $(0.91, 4)$ is negative and very steep. ◀

Practice 5. Graph $(x, y) = (3\cos(t), 2\sin(t))$ and find the slope of the tangent line when $t = \frac{\pi}{2}$.

When we calculated $\frac{dy}{dx}$, the slope of the tangent line to the graph of $(x(t), y(t))$, we used the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Each of these also has a geometric meaning: $\frac{dx}{dt}$ measures the rate of change of $x(t)$ with respect to t: it tells us whether the x-coordinate is increasing or decreasing as the t-variable increases (and how fast it is changing), while $\frac{dy}{dt}$ measures the rate of change of $y(t)$ with respect to t.



Example 9. For the parametric graph in the margin, determine whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive or negative when $t = 2$.

Solution. As we move through the point B (where $t = 2$) in the direction of increasing values of t, we are moving to the left, so $x(t)$ is decreasing and $\frac{dx}{dt} < 0$. The values of $y(t)$ are increasing, so $\frac{dy}{dt} > 0$. Finally, the slope of the tangent line, $\frac{dy}{dx}$, is negative. ◀

As a check on the sign of $\frac{dy}{dx}$ in the previous example:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\text{positive}}{\text{negative}} = \text{negative}$$

Practice 6. For the parametric graph in the previous example, tell whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive or negative at $t = 1$ and $t = 3$.

Speed

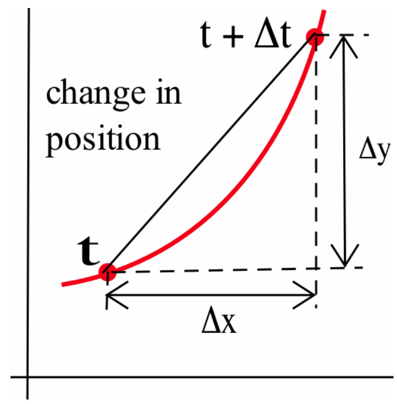
If we know the position of an object at any time, then we can determine its speed. The formula for speed comes from the distance formula and looks a lot like it, but involves derivatives.

If	$x = x(t)$ and $y = y(t)$ give the location of an object at time t and both are differentiable functions of t
then	the speed of the object is $$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Proof. The speed of an object is the limit, as $\Delta t \to 0$, of (see margin):

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} = \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta t)^2}}$$
$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \to \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

as $\Delta t \to 0$. □



Example 10. Find the speed of the object whose location at time t is $(x, y) = (\sin(t), t^2)$ when $t = 0$ and $t = 1$.

Solution. $\frac{dx}{dt} = \cos(t)$ and $\frac{dy}{dt} = 2t$ so:

$$\text{speed} = \sqrt{(\cos(t))^2 + (2t)^2} = \sqrt{\cos^2(t) + 4t^2}$$

When $t = 0$, speed $= \sqrt{\cos^2(0) + 4(0)^2} = \sqrt{1+0} = 1$. When $t = 1$, speed $= \sqrt{\cos^2(1) + 4(1)^2} \approx \sqrt{0.29 + 4} \approx 2.07$. ◀

Practice 7. Show that an object located at $(x, y) = (3\sin(t), 3\cos(t))$ at time t has a constant speed. (This object is moving on a circular path.)

Practice 8. Is the object at $(x, y) = (3\cos(t), 2\sin(t))$ at time t traveling faster at the top of the ellipse ($t = \frac{\pi}{2}$) or at the right edge ($t = 0$)?

2.5 Problems

In Problems 1–27, differentiate the given function.

1. $\ln(5x)$
2. $\ln(x^2)$
3. $\ln(x^k)$
4. $\ln(x^x) = x \cdot \ln(x)$
5. $\ln(\cos(x))$
6. $\cos(\ln(x))$
7. $\log_2(5x)$
8. $\log_2(kx)$
9. $\ln(\sin(x))$
10. $\ln(kx)$
11. $\log_2(\sin(x))$
12. $\ln(e^x)$
13. $\log_5(5^x)$
14. $\ln\left(e^{f(x)}\right)$
15. $x \cdot \ln(3x)$
16. $e^x \cdot \ln(x)$
17. $\frac{\ln(x)}{x}$
18. $\sqrt{x + \ln(3x)}$

19. $\ln\left(\sqrt{5x-3}\right)$

20. $\ln(\cos(t))$

21. $\cos(\ln(w))$

22. $\ln(ax+b)$

23. $\ln\left(\sqrt{t+1}\right)$

24. 3^x

25. $5^{\sin(x)}$

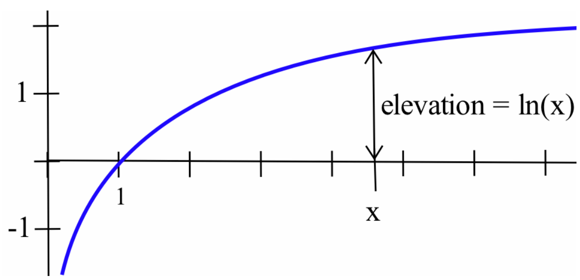
26. $x \cdot \ln(x) - x$

27. $\ln(\sec(x)+\tan(x))$

28. Find the slope of the line tangent to $f(x) = \ln(x)$ at the point $(e, 1)$. Find the slope of the line tangent to $g(x) = e^x$ at the point $(1, e)$. How are the slopes of f and g at these points related?

29. Find a point P on the graph of $f(x) = \ln(x)$ so the tangent line to f at P goes through the origin.

30. You are moving from left to right along the graph of $y = \ln(x)$ (see figure below).
    (a) If the x-coordinate of your location at time t seconds is $x(t) = 3t+2$, then how fast is your elevation increasing?
    (b) If the x-coordinate of your location at time t seconds is $x(t) = e^t$, then how fast is your elevation increasing?



31. The percent of a population, $p(t)$, who have heard a rumor by time t is often modeled by $p(t) = \dfrac{100}{1+Ae^{-t}} = 100\left(1+Ae^{-t}\right)^{-1}$ for some positive constant A. Calculate $p'(t)$, the rate at which the rumor is spreading.

32. If we start with A atoms of a radioactive material that has a "half-life" (the time it takes for half of the material to decay) of 500 years, then the number of radioactive atoms left after t years is $r(t) = A \cdot e^{-Kt}$ where $K = \dfrac{\ln(2)}{500}$. Calculate $r'(t)$ and show that $r'(t)$ is proportional to $r(t)$ (that is, $r'(t) = b \cdot r(t)$ for some constant b).

In 33–41, find a function with the given derivative.

33. $f'(x) = \dfrac{8}{x}$

34. $h'(x) = \dfrac{3}{3x+5}$

35. $f'(x) = \dfrac{\cos(x)}{3+\sin(x)}$

36. $g'(x) = \dfrac{x}{1+x^2}$

37. $g'(x) = 3e^{5x}$

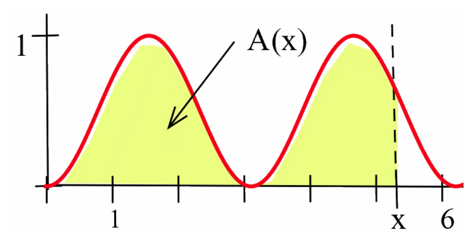
38. $h'(x) = e^2$

39. $f'(x) = 2x \cdot e^{x^2}$

40. $g'(x) = \cos(x)e^{\sin(x)}$

41. $h'(x) = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$

42. Define $A(x)$ to be the **area** bounded between the t-axis, the graph of $y = f(t)$ and a vertical line at $t = x$ (see figure below). The area under each "hump" of f is 2 square inches.
    (a) Graph $A(x)$ for $0 \le x \le 9$.
    (b) Graph $A'(x)$ for $0 \le x \le 9$.

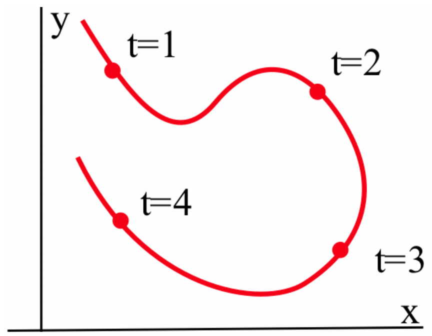


Problems 43–48 involve parametric equations.

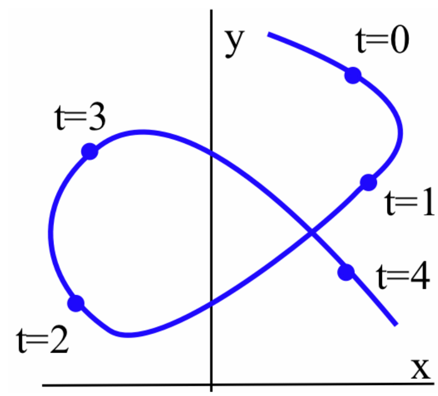
43. At time t minutes, robot A is at $(t, 2t+1)$ and robot B is at $(t^2, 2t^2+1)$.
    (a) Where is each robot when $t = 0$ and $t = 1$?
    (b) Sketch the path each robot follows during the first minute.
    (c) Find the slope of the tangent line, $\dfrac{dy}{dx}$, to the path of each robot at $t = 1$ minute.
    (d) Find the speed of each robot at $t = 1$ minute.
    (e) Discuss the motion of a robot that follows the path $(\sin(t), 2\sin(t)+1)$ for 20 minutes.

44. Let $x(t) = t+1$ and $y(t) = t^2$.
    (a) Graph $(x(t), y(t))$ for $-1 \le t \le 4$.
    (b) Find $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, the tangent slope $\dfrac{dy}{dx}$, and speed when $t = 1$ and $t = 4$.

45. For the parametric graph shown below, determine whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive, negative or 0 when $t = 1$ and $t = 3$.



46. For the parametric graph shown below, determine whether $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dy}{dx}$ are positive, negative or 0 when $t = 1$ and $t = 3$.

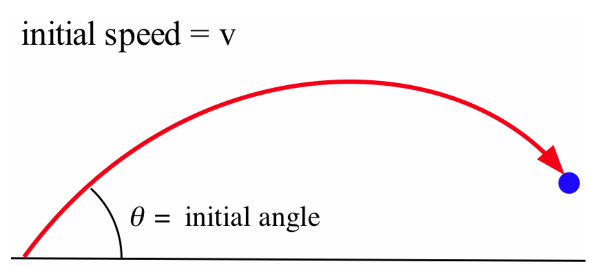


47. The parametric graph $(x(t), y(t))$ defined by $x(t) = R \cdot (t - \sin(t))$ and $y(t) = R \cdot (1 - \cos(t))$ is called a **cycloid**, the path of a light attached to the edge of a rolling wheel with radius R.
    (a) Graph $(x(t), y(t))$ for $0 \le t \le 4\pi$.
    (b) Find $\frac{dx}{dt}$, $\frac{dy}{dt}$, the tangent slope $\frac{dy}{dx}$, and speed when $t = \frac{\pi}{2}$ and $t = \pi$.

48. Describe the motion of particles whose locations at time t are $(\cos(t), \sin(t))$ and $(\cos(t), -\sin(t))$.

49. (a) Describe the path of a robot whose location at time t is $(3 \cdot \cos(t), 5 \cdot \sin(t))$.
    (b) Describe the path of a robot whose location at time t is $(A \cdot \cos(t), B \cdot \sin(t))$.
    (c) Give parametric equations so the robot will move along the same path as in part (a) but in the opposite direction.

50. After t seconds, a projectile hurled with initial velocity v and angle θ will be at $x(t) = v \cdot \cos(\theta) \cdot t$ feet and $y(t) = v \cdot \sin(\theta) \cdot t - 16t^2$ feet (see figure below). (This formula neglects air resistance.)
    (a) For an initial velocity of 80 feet/second and an angle of $\frac{\pi}{4}$, find $T > 0$ so that $y(T) = 0$. What does this value for t represent physically? Evaluate $x(T)$.
    (b) For v and θ in part (a), calculate $\frac{dy}{dx}$. Find T so that $\frac{dy}{dx} = 0$ at $t = T$, and evaluate $x(T)$. What does $x(T)$ represent physically?
    (c) What initial velocity is needed so a ball hit at an angle of $\frac{\pi}{4} \approx 0.7854$ will go over a 40-foot-high fence 350 feet away?
    (d) What initial velocity is needed so a ball hit at an angle of 0.7 radians will go over a 40-foot-high fence 350 feet away?



51. Use the method from the proof that $\mathbf{D}(\ln(x)) = \frac{1}{x}$ to compute the derivative $\mathbf{D}(\arctan(x))$:
    (a) Rewrite $y = \arctan(x)$ as $\tan(y) = x$.
    (b) Differentiate both sides using the Chain Rule and solve for y'.
    (c) Use the identity $1 + \tan^2(\theta) = \sec^2(\theta)$ and the fact that $\tan(y) = x$ to show that $y' = \frac{1}{1+x^2}$.

52. Use the method from the proof that $\mathbf{D}(\ln(x)) = \frac{1}{x}$ to compute the derivative $\mathbf{D}(\arcsin(x))$:
    (a) Rewrite $y = \arcsin(x)$ as $\sin(y) = x$.
    (b) Differentiate both sides using the Chain Rule and solve for y'.
    (c) Use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ and the fact that $\sin(y) = x$ to show that $y' = \frac{1}{\sqrt{1-x^2}}$.

2.5 Practice Answers

1. $\log_9(20) = \dfrac{\log(20)}{\log(9)} \approx 1.3634165 \approx \dfrac{\ln(20)}{\ln(9)}$

   $\log_3(20) = \dfrac{\log(20)}{\log(3)} \approx 2.726833 \approx \dfrac{\ln(20)}{\ln(3)}$

   $\log_\pi(e) = \dfrac{\log(e)}{\log(\pi)} \approx 0.8735685 \approx \dfrac{\ln(e)}{\ln(\pi)} = \dfrac{1}{\ln(\pi)}$

2. $\mathbf{D}\left(\log_{10}(\sin(x))\right) = \dfrac{1}{\sin(x)\cdot\ln(10)}\mathbf{D}(\sin(x)) = \dfrac{\cos(x)}{\sin(x)\cdot\ln(10)}$

   $\mathbf{D}\left(\log_\pi(e^x)\right) = \dfrac{1}{e^x\cdot\ln(\pi)}\mathbf{D}(e^x) = \dfrac{e^x}{e^x\cdot\ln(\pi)} = \dfrac{1}{\ln(\pi)}$

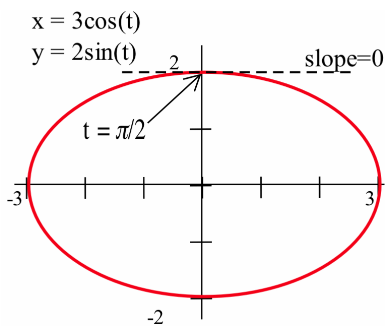
3. $\mathbf{D}\left(\sin(2^x)\right) = \cos(2^x)\,\mathbf{D}\left(2^x\right) = \cos(2^x)\cdot 2^x\cdot\ln(2)$

   $\dfrac{d}{dt}\left(3^{t^2}\right) = 3^{t^2}\ln(3)\,\mathbf{D}(t^2) = 3^{t^2}\ln(3)\cdot 2t$

4. $T = \dfrac{72}{1+h} = \dfrac{72}{1+t+\sin(t)} \Rightarrow$

   $\dfrac{dT}{dt} = \dfrac{(1+t+\sin(t))\cdot 0 - 72\cdot\mathbf{D}(1+t+\sin(t))}{(1+t+\sin(t))^2} = \dfrac{-72(1+\cos(t))}{(1+t+\sin(t))^2}$

   When $t=5$, $\dfrac{dT}{dt} = \dfrac{-72(1+\cos(5))}{(1+5+\sin(5))^2} \approx -3.63695$.



5. $x(t) = 3\cos(t) \Rightarrow \dfrac{dx}{dt} = -3\sin(t)$, $y(t) = 2\sin(t) \Rightarrow \frac{dy}{dt} = 2\cos(t)$:

   $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos(t)}{-3\sin(t)} \Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{2}} = \frac{2\cos(\frac{\pi}{2})}{-3\sin(\frac{\pi}{2})} = \frac{2\cdot 0}{-3\cdot 1} = 0$$

   (See margin for graph.)

6. $x=1$: positive, positive, positive. $x=3$: positive, negative, negative.

7. $x(t) = 3\sin(t) \Rightarrow \frac{dx}{dt} = 3\cos(t)$ and $y(t) = 3\cos(t) \Rightarrow \frac{dy}{dt} = -3\sin(t)$. So:

   $$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3\cos(t))^2 + (-3\sin(t))^2}$$
   $$= \sqrt{9\cdot\cos^2(t) + 9\cdot\sin^2(t)} = \sqrt{9} = 3 \quad \text{(a constant)}$$

8. $x(t) = 3\cos(t) \Rightarrow \frac{dx}{dt} = -3\sin(t)$ and $y(t) = 2\sin(t) \Rightarrow \frac{dy}{dt} = 2\cos(t)$ so:

   $$\text{speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-3\sin(t))^2 + (2\cos(t))^2}$$
   $$= \sqrt{9\cdot\sin^2(t) + 4\cdot\cos^2(t)}$$

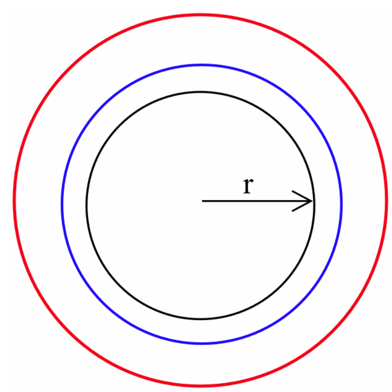
   When $t=0$, the speed is $\sqrt{9\cdot 0^2 + 4\cdot 1^2} = 2$.
   When $t = \frac{\pi}{2}$, the speed is $\sqrt{9\cdot 1^2 + 4\cdot 0^2} = 3$ (faster).

2.6 Related Rates

Throughout the next several sections we'll look at a variety of applications of derivatives. Probably no single application will be of interest or use to everyone, but at least some of them should be useful to you. Applications also reinforce what you have been practicing: they require that you recall what a derivative means and require you to use the differentiation techniques covered in the last several sections. Most people gain a deeper understanding and appreciation of a tool as they use it, and differentiation is both a powerful concept and a useful tool.

The Derivative as a Rate of Change

In Section 2.1, we discussed several interpretations of the derivative of a function. Here we will examine the "rate of change of a function" interpretation. If several variables or quantities are related to each other and some of the variables are changing at a known rate, then we can use derivatives to determine how rapidly the other variables must be changing.



Example 1. The radius of a circle is increasing at a rate of 10 feet each second (see margin figure) and we want to know how fast the **area** of the circle is increasing when the radius is 5 feet. What can we do?

Solution. We could get an *approximate* answer by calculating the area of the circle when the radius is 5 feet:

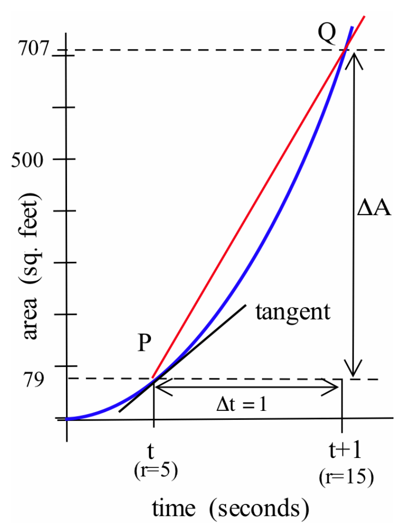
$$A = \pi r^2 = \pi(5 \text{ feet})^2 \approx 78.6 \text{ feet}^2$$

and the area 1 second later when the radius is 10 feet larger than before:

$$A = \pi r^2 = \pi(15 \text{ feet})^2 \approx 706.9 \text{ feet}^2$$

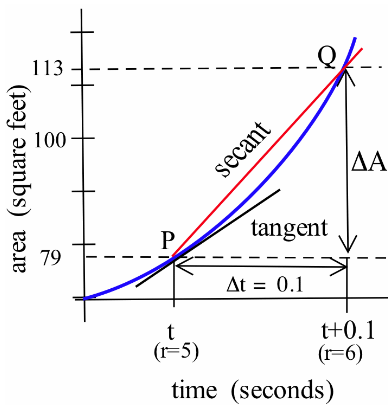
and then computing:

$$\frac{\Delta\text{area}}{\Delta\text{time}} = \frac{706.9 \text{ feet}^2 - 78.6 \text{ feet}^2}{1 \text{ second}} = 628.3 \frac{\text{ft}^2}{\text{sec}}$$



This approximate answer represents the average change in area during the 1-second period when the radius increased from 5 feet to 15 feet. It is also the slope of the secant line through the points P and Q in the margin figure, and it is clearly not a very good approximation of the instantaneous rate of change of the area, the slope of the tangent line at the point P.

We could get a *better approximation* by calculating $\frac{\Delta A}{\Delta t}$ over a shorter time interval, say $\Delta t = 0.1$ seconds. In this scenario, the original area



is still 78.6 ft^2 but the new area (after $t = 0.1$ seconds has passed) is $A = \pi(6 \text{ feet})^2 \approx 113.1 \text{ ft}^2$ (why is the new radius 6 feet?) so:

$$\frac{\Delta A}{\Delta t} = \frac{113.1 \text{ feet}^2 - 78.6 \text{ feet}^2}{0.1 \text{ second}} = 345 \frac{\text{ft}^2}{\text{sec}}$$

This is the slope of the secant line through the points P and Q in the margin figure, which represents a much better approximation of the slope of the tangent line at P—but it is still only an approximation. Using derivatives, we can get an **exact** answer without doing very much work at all.

We know that the two variables in this problem, the radius r and the area A, are related to each other by the formula $A = \pi r^2$. We also know that both r and A are changing over time, so each of them is a function of an additional variable t (time, in seconds): $r(t)$ and $A(t)$.

We want to know the rate of change of the area "when the radius is 5 feet" so if $t = 0$ corresponds to the particular moment in time when the radius is 5 feet, we can write $r(0) = 5$.

The statement that "the radius is increasing at a rate of 10 feet each second" can be translated into a mathematical statement about the rate of change, the derivative of r (radius) with respect to t (time): if $t = 0$ corresponds to the moment when the radius is 5 feet, then $r'(0) = \frac{dr}{dt} = 10$ ft/sec.

The question about the rate of change of the area is a question about $A'(t) = \frac{dA}{dt}$.

Collecting all of this information...

- **variables**: $r(t)$ = radius at time t, $A(t)$ = area at time t
- **we know**: $r(0) = 5$ feet and $r'(0) = 10$ ft/sec
- **we want to know**: $A'(0)$, the rate of change of area with respect to time at the moment when $r = 5$ feet
- **connecting equation**: $A = \pi r^2$ or $A(t) = \pi [r(t)]^2$

To find $A'(0)$ we must first find $A'(t)$ and then evaluate this derivative at $t = 0$. Differentiating both sides of the connecting equation, we get:

$$A(t) = \pi [r(t)]^2 \Rightarrow A'(t) = 2\pi [r(t)]^1 \cdot r'(t) \Rightarrow A'(t) = 2\pi \cdot r(t) \cdot r'(t)$$

Notice that we have used the Power Rule for Functions (or, more generally, the Chain Rule) because the area is a function of the radius, which is a function of time.

Now we can plug in $t = 0$ and use the information we know:

$$A'(0) = 2\pi \cdot r(0) \cdot r'(0) = 2\pi \cdot 5 \cdot 10 = 100\pi$$

When the radius is 5 feet, the area is increasing at 100π $\text{ft}^2/\text{sec} \approx 314.2$ square feet per second. ◀

Before considering other examples, let's review the solution to the previous example. The statement "the radius is increasing at a rate of 10 feet each second" implies that this rate of change is the same at $t = 0$ (the moment in time we were interested in) as at any other time during this process, say $t = 1.5$ or $t = 98$: $r'(0) = r'(1.5) = r'(98) = 10$. But we only used the fact that $r'(0) = 10$ in our solution.

We should take care in future problems to consider whether the information we are given about rates of change holds true all the time or just at a particular moment in time. That didn't matter in our first example, but it might in other situations.

Next, notice that we let $t = 0$ correspond to the particular moment in time the question asked about (the moment when $r = 5$). But this choice was arbitrary: we could have let this moment correspond to $t = 2.8$ or $t = 7\pi$ and the eventual answer would have been the same.

Finally, notice that we explicitly wrote each variable (and their derivatives) as a function of the time variable, t: $A(t)$, $r(t)$, $A'(t)$ and $r'(t)$. Consequently, we used the composition form of the Chain Rule:

$$(A \circ r)'(t) = A'(r(t)) \cdot r'(t)$$

Let's redo the previous example using the Leibniz form of the Chain Rule, keeping the above observations in mind.

Solution. We know that the two variables in this problem, the radius r and the area A, are related to each other by the formula $A = \pi r^2$. We also know that both r and A are changing over time, so each of them is a function of an additional variable t (time, in seconds).

We want to know the rate of change of the area "when the radius is 5 feet," which translates to evaluating $\frac{dA}{dt}$ at the moment when $r = 5$. We write this in Leibniz notation as:

$$\left.\frac{dA}{dt}\right|_{r=5}$$

The statement that "the radius is increasing at a rate of 10 feet each second" translates into $\frac{dr}{dt} = 10$. From the connecting equation $A = \pi r^2$ we know that $\frac{dA}{dr} = 2\pi r$. Furthermore,the Chain Rule tells us that:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

We know that $\frac{dA}{dr} = 2\pi r$ and $\frac{dr}{dt} = 10$ are *always* true, so we can rewrite the Chain Rule statement above as:

$$\frac{dA}{dt} = 2\pi r \cdot 10 = 20\pi r$$

Finally, we evaluate both sides at the moment in time we are interested in (the moment when $r = 5$):

$$\left.\frac{dA}{dt}\right|_{r=5} = 20\pi r\Big|_{r=5} = 20\pi \cdot 5 = 100\pi \approx 314.2$$

which is the same answer we found in the original solution. ◀

The key steps in finding the rate of change of the area of the circle were:

- write the known information in a mathematical form, expressing rates of change as derivatives: $\frac{dr}{dt} = 10$ ft/sec
- write the question in a mathematical form: $\frac{dA}{dt} = ?$
- find an equation connecting or relating the variables: $A = \pi r^2$
- differentiate both sides of the connecting equation using the Chain Rule (and other differentiation patterns as necessary): $\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$
- substitute all of the known values that are *always* true into the equation resulting from the previous step and (if necessary) solve for the desired quantity in the resulting equation: $\frac{dA}{dt} = 2\pi r \cdot 10$
- substitute all of the known values that are true at the particular moment in time the question asks about into the equation resulting from the previous step: $\left.\frac{dA}{dt}\right|_{r=5} = \left.2\pi r \cdot 10\right|_{r=5} = 100\pi$

Example 2. Divers' lives depend on understanding situations involving related rates. In water, the pressure at a depth of x feet is approximately $P(x) = 15\left(1 + \frac{x}{33}\right)$ pounds per square inch (compared to approximately $P(0) = 15$ pounds per square inch at sea level). Volume is inversely proportional to the pressure, $V = \frac{k}{P}$, so doubling the pressure will result in half the original volume. Remember that volume is a function of the pressure: $V = V(P)$.

(a) Suppose a diver's lungs, at a depth of 66 feet, contained 1 cubic foot of air and the diver ascended to the surface without releasing any air. What would happen?

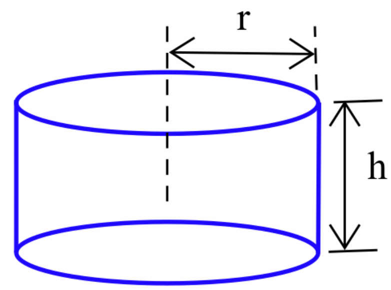
(b) If a diver started at a depth of 66 feet and ascended at a rate of 2 feet per second, how fast would the pressure be changing?

(Dives deeper than 50 feet also involve a risk of the "bends," or decompression sickness, if the ascent is too rapid. Tables are available that show the safe rates of ascent from different depths.)

Solution. (a) The diver would risk rupturing his or her lungs. The 1 cubic foot of air at a depth of 66 feet would be at a pressure of $P(66) = 15\left(1 + \frac{66}{33}\right) = 45$ pounds per square inch (psi). Because the pressure at sea level, $P(0) = 15$ psi, is only $\frac{1}{3}$ as great, each cubic foot of air would expand to 3 cubic feet, and the diver's lungs would be in danger. Divers are taught to release air as they ascend to avoid this danger. (b) The diver is ascending at a rate of 2 feet/second so the rate of change of the diver's depth with respect to time is $\frac{dx}{dt} = -2$ ft/s. (Why is this rate of change negative?) The pressure is $P = 15\left(1 + \frac{x}{33}\right) = 15 + \frac{15}{33}x$, a function of x, so using the Chain Rule:

$$\frac{dP}{dt} = \frac{dP}{dx} \cdot \frac{dx}{dt} = \frac{15}{33}\,\frac{\text{psi}}{\text{ft}} \cdot \left(-2\,\frac{\text{ft}}{\text{sec}}\right) = -\frac{30}{33}\,\frac{\text{psi}}{\text{sec}} \approx -0.91\,\frac{\text{psi}}{\text{sec}}$$

The rates of change in this problem are constant (they hold true at any moment in time during the ascent) so we are done. ◀



Example 3. The height of a cylinder is increasing at 7 meters per second and the radius is increasing at 3 meters per second. How fast is the volume changing when the cylinder is 5 meters high and has a radius of 6 meters? (See margin.)

Solution. First we need to translate our known information into a mathematical format. The height and radius are given: at the particular moment in time the question asks about, h = height = 5 m and r = radius = 6 m. We are also told how fast h and r are changing at this moment in time: $\frac{dh}{dt} = 7$ m/sec and $\frac{dr}{dt} = 3$ m/sec. Finally, we are asked to find $\frac{dV}{dt}$, and we should expect the units of $\frac{dV}{dt}$ to be the same as $\frac{\Delta V}{\Delta t}$, which are m^3/sec.

- **variables**: $h(t)$ = height at time t seconds, $r(t)$ = radius at time t, $V(t)$ = volume at time t.
- **we know**: at a particular moment in time, $h = 5$ m, $\frac{dh}{dt} = 7$ m/sec, $r = 6$ m and $\frac{dr}{dt} = 3$ m/sec
- **we want to know**: $\frac{dV}{dt}$ at this particular moment in time

We also need an equation that relates the variables h, r and V (all of which are functions of time t) to each other:

- **connecting equation**: $V = \pi r^2 h$

Differentiating each side of this equation with respect to t (remembering that h, r and V are functions of t), we have:

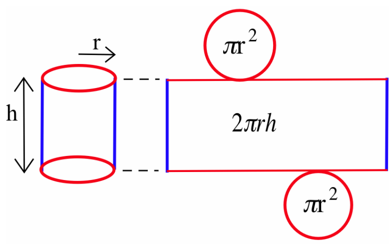
$$\begin{aligned}\frac{dV}{dt} = \frac{d}{dt}\left(\pi r^2 h\right) &= \pi r^2 \cdot \frac{dh}{dt} + h \cdot \frac{d}{dt}\left(\pi r^2\right) \\ &= \pi r^2 \cdot \frac{dh}{dt} + h \cdot 2\pi r \cdot \frac{dr}{dt}\end{aligned}$$

using the Product Rule (on the product $\pi r^2 \cdot h$) and the Power Rule for Functions (on πr^2, remembering that r is actually a function of t).

The rest of the solution just involves substituting values and doing some arithmetic. At the particular moment in time we're interested in:

$$\begin{aligned}\frac{dV}{dt} &= \pi \cdot 6^2\,\text{m}^2 \cdot 7\,\frac{\text{m}}{\text{sec}} + 5\,\text{m} \cdot 2\pi \cdot 6\,\text{m} \cdot 3\,\frac{\text{m}}{\text{sec}} \\ &= 432\pi\,\frac{\text{m}^3}{\text{sec}} \approx 1357.2\,\frac{\text{m}^3}{\text{sec}}\end{aligned}$$

The volume of the cylinder is increasing at a rate of 1,357.2 cubic meters per second. (It is always encouraging when the units of our answer are the ones we expect.) ◀

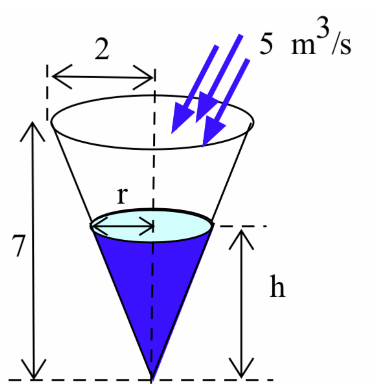


Practice 1. How fast is the **surface area** of the cylinder changing in the previous example? (Assume that h, r, $\frac{dh}{dt}$ and $\frac{dr}{dt}$ have the same values as in the example and use the figure in the margin to help you determine an equation relating the surface area of the cylinder to the variables h and r. The cylinder includes a top and bottom.)

Practice 2. How fast is the **volume** of the cylinder in the previous example changing if the radius is decreasing at a rate of 3 meters per second? (The height, radius and rate of change of the height are the same as in the previous example: 5 m, 6 m and 7 m/sec respectively.)

Usually, the most difficult part of Related Rates problems is to find an equation that relates or connects all of the variables. In the previous problems, the relating equations required a knowledge of geometry and formulas for areas and volumes (or knowing where to look them up). Other Related Rates problems may require information about similar triangles, the Pythagorean Theorem or trigonometric identities: the information required varies from problem to problem.

It is a good idea—a very good idea—to draw a picture of the physical situation whenever possible. It is also a good idea, particularly if the problem is very important (your next raise depends on getting the right answer), to calculate at least one *approximate* answer as a check of your exact answer.



Example 4. Water is flowing into a conical tank at a rate of 5 m^3/sec. If the radius of the top of the cone is 2 m, the height is 7 m, and the depth of the water is 4 m, then how fast is the water level rising?

Solution. Let's define our variables to be h = height (or depth) of the water in the cone and V = the volume of the water in the cone. Both h and V are changing, and both of them are functions of time t. We are told in the problem that $h = 4$ m and $\frac{dV}{dt} = 5 \text{ m}^3/\text{sec}$, and we are asked to find $\frac{dh}{dt}$. We expect that the units of $\frac{dh}{dt}$ will be the same as $\frac{\Delta h}{\Delta t}$, which are meters/second.

- **variables**: $h(t)$ = height at time t seconds, $r(t)$ = radius of the top surface of the water at time t, $V(t)$ = volume of water at time t
- **we know**: $\frac{dV}{dt} = 5 \text{ m}^3/\text{sec}$ (always true) and $h = 4$ m (at a particular moment)
- **we want to know**: $\frac{dh}{dt}$ at this particular moment

Unfortunately, the equation for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, also involves an additional variable r, the radius of the cone at the top of the water. This is a situation in which a picture can be a great help by suggesting that we have a pair of similar triangles:

$$\frac{r}{h} = \frac{\text{top radius}}{\text{total height}} = \frac{2 \text{ m}}{7 \text{ m}} = \frac{2}{7} \quad \Rightarrow \quad r = \frac{2}{7}h$$

Knowing this, we can rewrite the volume of the water contained in the cone, $V = \frac{1}{3}\pi r^2 h$, as a function of the single variable h:

- **connecting equation**: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{2}{7}h\right)^2 h = \frac{4}{147}\pi h^3$

The rest of the solution is reasonably straightforward.

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = \frac{d}{dh}\left(\frac{4}{147}\pi h^3\right) \cdot \frac{dh}{dt}$$

We know $\frac{dV}{dt} = 5$ always holds, and the derivative is easy to compute:

$$5 = \frac{4}{49}\pi h^2 \cdot \frac{dh}{dt}$$

At the particular moment in time we want to know about (when $h = 4$):

$$5 = \frac{4}{49}\pi h^2\Big|_{h=4} \cdot \frac{dh}{dt}\Big|_{h=4} \quad \Rightarrow \quad 5 = \frac{64\pi}{49} \cdot \frac{dh}{dt}\Big|_{h=4}$$

and we can now solve for the quantity of interest:

$$\frac{dh}{dt}\Big|_{h=4} = \frac{5}{\frac{64\pi}{49}} = \frac{245}{64\pi} \approx 1.22 \, \frac{\text{m}}{\text{sec}}$$

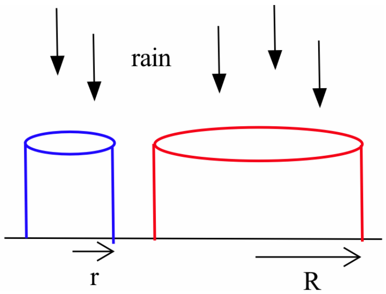
This example was a bit more challenging because we needed to use similar triangles to get an equation relating V to h and because we eventually needed to do some arithmetic to solve for $\frac{dh}{dt}$. ◀

Practice 3. A rainbow trout has taken the fly at the end of a 60-foot fishing line, and the line is being reeled in at a rate of 30 feet per minute. If the tip of the rod is 10 feet above the water and the trout is at the surface of the water, how fast is the trout being pulled toward the angler? (Hint: Draw a picture and use the Pythagorean Theorem.)

Example 5. When rain is falling vertically, the amount (volume) of rain collected in a cylinder is proportional to the area of the **opening** of the cylinder. If you place a narrow cylindrical glass and a wide cylindrical glass out in the rain:

(a) which glass will collect water faster?

(b) in which glass will the water level rise faster?



Solution. Let's assume that the smaller glass has a radius of r and the larger glass has a radius of R, so that $R > r$. The areas of their openings are πr^2 and πR^2, respectively. Call the volume of water collected in each glass v (for the smaller glass) and V (for the larger glass).

(a) The smaller glass will collect water at the rate $\frac{dv}{dt} = K \cdot \pi r^2$ and the larger at the rate $\frac{dV}{dt} = K \cdot \pi R^2$ so $\frac{dV}{dt} > \frac{dv}{dt}$ and the larger glass will collect water faster than the smaller glass.

(b) The volume of water in each glass is a function of the radius of the glass and the height of the water in the glass: $v = \pi r^2 h$ and $V = \pi R^2 H$ where h and H are the heights of the water levels in the smaller and larger glasses, respectively. The heights h and H vary with t (in other words, they are each functions of t) while the radii (r and R) remain constant, so:

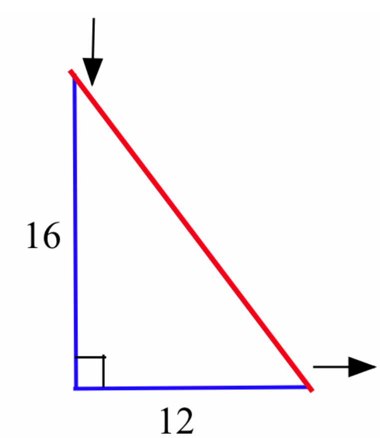
$$\frac{dv}{dt} = \frac{d}{dt}\left(\pi r^2 h\right) = \pi r^2 \frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{\frac{dv}{dt}}{\pi r^2} = \frac{K\pi r^2}{\pi r^2} = K$$

Similarly:

$$\frac{dV}{dt} = \frac{d}{dt}\left(\pi R^2 H\right) = \pi R^2 \frac{dH}{dt} \Rightarrow \frac{dH}{dt} = \frac{\frac{dV}{dt}}{\pi R^2} = \frac{K\pi R^2}{\pi R^2} = K$$

So $\frac{dh}{dt} = K = \frac{dH}{dt}$, which tells us the water level in each glass is rising at the same rate. In a one-minute period, the larger glass will collect more rain, but the larger glass also requires more rain to raise its water level by a fixed amount. How do you think the volumes and water levels would change if we placed a small glass and a large plastic (rectangular) box side by side in the rain? ◀

2.6 Problems



1. An expandable sphere is being filled with liquid at a constant rate from a tap (imagine a water balloon connected to a faucet). When the radius of the sphere is 3 inches, the radius is increasing at 2 inches per minute. How fast is the liquid coming out of the tap? ($V = \frac{4}{3}\pi r^3$)

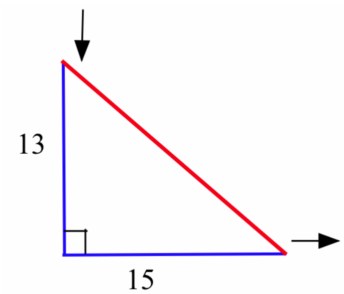
2. The 12-inch base of a right triangle is growing at 3 inches per hour, and the 16-inch height of the triangle is shrinking at 3 inches per hour (see figure in the margin).

   (a) Is the area increasing or decreasing?

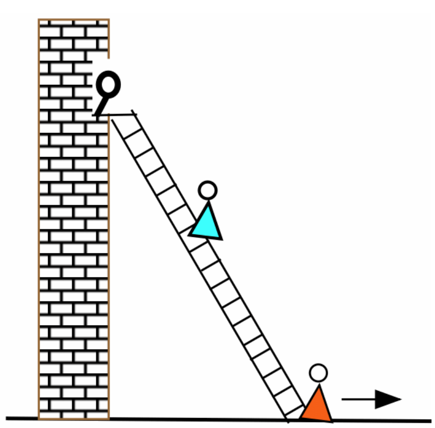
   (b) Is the perimeter increasing or decreasing?

   (c) Is the hypotenuse increasing or decreasing?

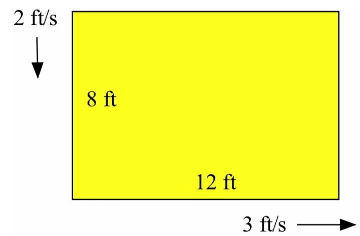
3. One hour later the right triangle in the previous problem is 15 inches long and 13 inches high (see figure below) and the base and height are changing at the same rate as in Problem 2.
   (a) Is the area increasing or decreasing now?
   (b) Is the hypotenuse increasing or decreasing?
   (c) Is the perimeter increasing or decreasing?



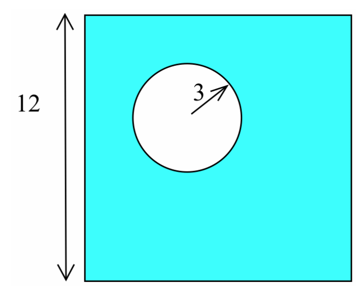
4. A young woman and her boyfriend plan to elope, but she must rescue him from his mother, who has locked him in his room. The young woman has placed a 20-foot long ladder against his house and is knocking on his window when his mother begins pulling the bottom of the ladder away from the house at a rate of 3 feet per second (see figure below). How fast is the top of the ladder (and the young couple) falling when the bottom of the ladder is:
   (a) 12 feet from the bottom of the wall?
   (b) 16 feet from the bottom of the wall?
   (c) 19 feet from the bottom of the wall?



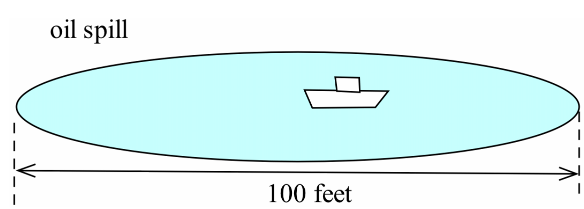
5. The length of a 12-foot by 8-foot rectangle is increasing at a rate of 3 feet per second and the width is decreasing at 2 feet per second (see figure below).
   (a) How fast is the perimeter changing?
   (b) How fast is the area changing?



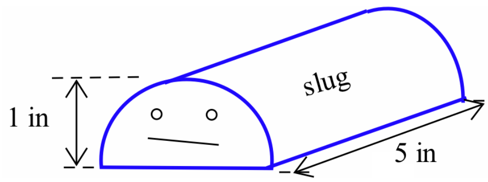
6. A circle of radius 3 inches is inside a square with 12-inch sides (see figure below). How fast is the area between the circle and square changing if the radius is increasing at 4 inches per minute and the sides are increasing at 2 inches per minute?



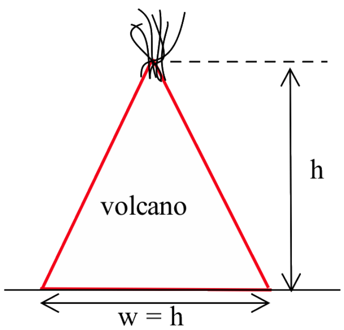
7. An oil tanker in Puget Sound has sprung a leak, and a circular oil slick is forming. The oil slick is 4 inches thick everywhere, is 100 feet in diameter, and the diameter is increasing at 12 feet per hour. Your job, as the Coast Guard commander or the tanker's captain, is to determine how fast the oil is leaking from the tanker.



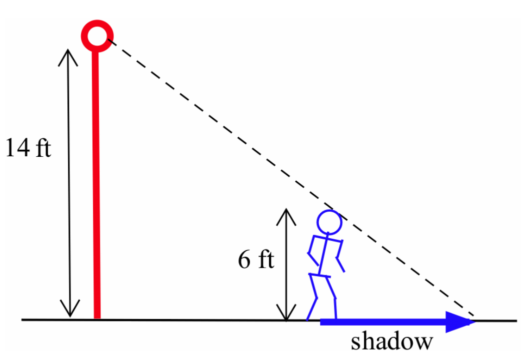
8. A mathematical species of slug has a semicircular cross section and is always 5 times as long as it is high (see figure below). When the slug is 5 inches long, it is growing at 0.2 inches per week.
   (a) How fast is its volume increasing?
   (b) How fast is the area of its "foot" (the part of the slug in contact with the ground) increasing?



9. Lava flowing from a hole at the top of a hill is forming a conical mountain whose height is always the same as the width of its base (see figure below). If the mountain is increasing in height at 2 feet per hour when it is 500 feet high, how fast is the lava flowing (that is, how fast is the volume of the mountain increasing)? ($V = \frac{1}{3}\pi r^2 h$)

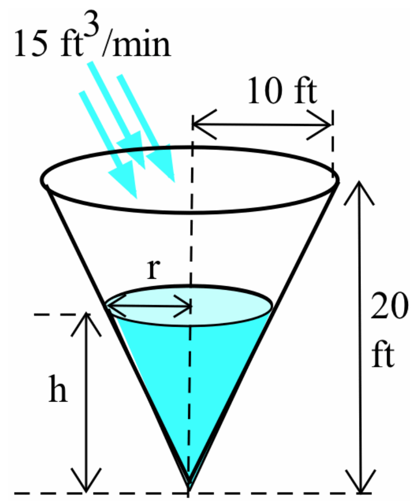


10. A 6-foot-tall person is walking away from a 14-foot lamp post at 3 feet per second. When the person is 10 feet away from the lamp post:
    (a) how fast is the length of the shadow changing?
    (b) how fast is the tip of the shadow moving away from the lamp post?

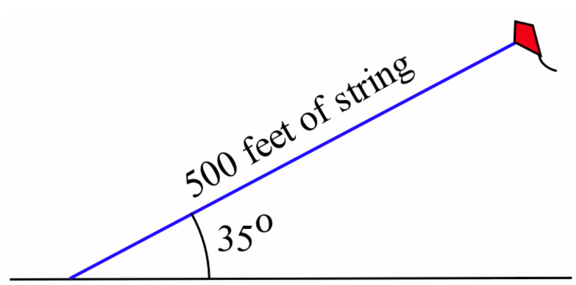


11. Redo the previous problem if the person is 20 feet from the lamp post.

12. Water is being poured at a rate of 15 cubic feet per minute into a conical reservoir that is 20 feet deep and has a top radius of 10 feet (see below).
    (a) How long will it take to fill the empty reservoir?
    (b) How fast is the water level rising when the water is 4 feet deep?
    (c) How fast is the water level rising when the water is 16 feet deep?



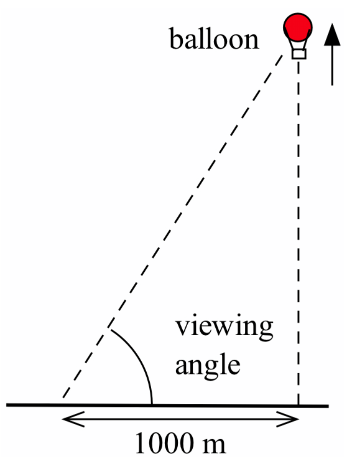
13. The string of a kite is perfectly taut and always makes an angle of 35° above horizontal.
    (a) If the kite flyer has let out 500 feet of string, how high is the kite?
    (b) If the string is let out at a rate of 10 feet per second, how fast is the kite's height increasing?



14. A small tracking telescope is viewing a hot-air balloon rise from a point 1,000 meters away from a point directly under the balloon.

    (a) When the viewing angle is $20°$, it is increasing at a rate of $3°$ per minute. How high is the balloon, and how fast is it rising?

    (b) When the viewing angle is $80°$, it is increasing at a rate of $2°$ per minute. How high is the balloon, and how fast is it rising?



15. The 8-foot diameter of a spherical gas bubble is increasing at 2 feet per hour, and the 12-foot-long edges of a cube containing the bubble are increasing at 3 feet per hour. Is the volume contained between the spherical bubble and the cube increasing or decreasing? At what rate?

16. In general, the strength S of an animal is proportional to the cross-sectional area of its muscles, and this area is proportional to the square of its height H, so the strength $S = aH^2$. Similarly, the weight W of the animal is proportional to the cube of its height, so $W = bH^3$. Finally, the relative strength R of an animal is the ratio of its strength to its weight. As the animal grows, show that its strength and weight increase, but that the relative strength decreases.

17. The snow in a hemispherical pile melts at a rate proportional to its exposed surface area (the surface area of the hemisphere). Show that the height of the snow pile is decreasing at a constant rate.

18. If the rate at which water vapor condenses onto a spherical raindrop is proportional to the surface area of the raindrop, show that the radius of the raindrop will increase at a constant rate.

19. Define $A(x)$ to be the area bounded by the t- and y-axes, and the lines $y = 5$ and $t = x$.

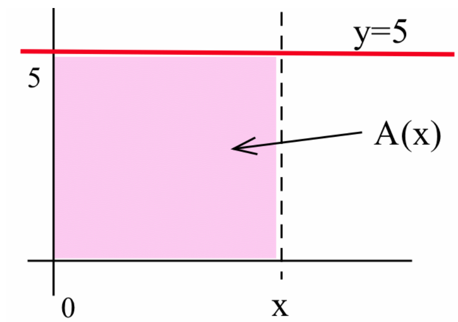
    (a) Find a formula for A as a function of x.

    (b) Determine $A'(x)$ when $x = 1, 2, 4$ and 9.

    (c) If x is a function of time, $x(t) = t^2$, find a formula for A as a function of t.

    (d) Determine $A'(t)$ when $t = 1, 2$ and 3.

    (e) Suppose instead $x(t) = 2 + \sin(t)$. Find a formula for $A(t)$ and determine $A'(t)$.



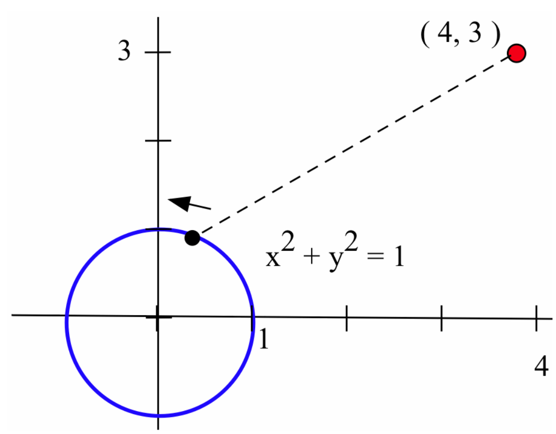
20. The point P is going around the circle $x^2 + y^2 = 1$ twice a minute. How fast is the distance between the point P and the point $(4, 3)$ changing:

    (a) when $P = (1, 0)$?

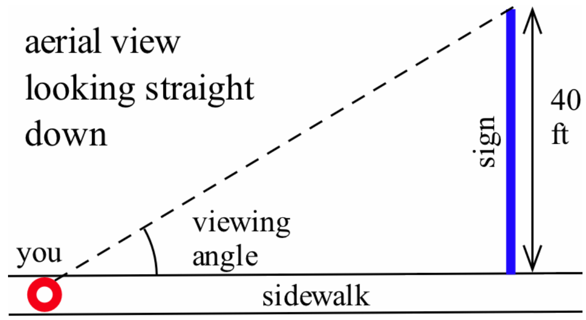
    (b) when $P = (0, 1)$?

    (c) when $P = (0.8, 0.6)$?

    (Suggestion: Write x and y as parametric functions of time t.)



21. You are walking along a sidewalk toward a 40-foot-wide sign adjacent to the sidewalk and perpendicular to it. If your viewing angle θ is $10°$:
    (a) how far are you from the corner of the sign?
    (b) how fast is your viewing angle changing if you are walking at 25 feet per minute?
    (c) how fast are you walking if the angle is increasing at $2°$ per minute?



2.6 Practice Answers

1. The surface area is $S = 2\pi rh + 2\pi r^2$. From the Example, we know that $\frac{dh}{dt} = 7$ m/sec and $\frac{dr}{dt} = 3$ m/sec, and we want to know how fast the surface area is changing when $h = 5$ m and $r = 6$ m.

$$\begin{aligned}\frac{dS}{dt} &= 2\pi r \cdot \frac{dh}{dt} + 2\pi \frac{dr}{dt} \cdot h + 2\pi \cdot 2r \cdot \frac{dr}{dt} \\ &= 2\pi (6\,\text{m}) \left(7\,\frac{\text{m}}{\text{sec}}\right) + 2\pi \left(3\frac{\text{m}}{\text{sec}}\right)(5m) + 2\pi\,(2 \cdot 6\,\text{m}) \left(3\,\frac{\text{m}}{\text{sec}}\right) \\ &= 186\pi\,\frac{\text{m}^2}{\text{sec}} \approx 584.34\,\frac{\text{m}^2}{\text{sec}}\end{aligned}$$

Note that the units represent a rate of change of area.

2. The volume is $V = \pi r^2 h$. We know that $\frac{dr}{dt} = -3$ m/sec and that $h = 5$ m, $r = 6$ m and $\frac{dh}{dt} = 7$ m/sec.

$$\begin{aligned}\frac{dV}{dt} &= \pi r^2 \cdot \frac{dh}{dt} + \pi \cdot 2r \cdot \frac{dr}{dt} \\ &= \pi (6\,\text{m})^2 \left(7\,\frac{\text{m}}{\text{sec}}\right) + \pi (2 \cdot 6\,\text{m}) \left(-3\,\frac{\text{m}}{\text{sec}}\right) \\ &= 72\pi\,\frac{\text{m}^3}{\text{sec}} \approx 226.19\,\frac{\text{m}^3}{\text{sec}}\end{aligned}$$

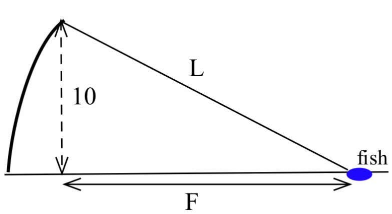
Note that the units represent a rate of change of volume.

3. See margin figure. We know $\frac{dL}{dt} = -30\,\frac{\text{ft}}{\text{min}}$ (always true); F represents the distance from the fish to a point directly below the tip of the rod, and the distance from that point to the angler remains constant, so $\frac{dF}{dt}$ will equal the rate at which the fish is moving toward the angler. We want to know $\left.\frac{dF}{dt}\right|_{L=60}$. The Pythagorean Theorem connects F and L: $F^2 + 10^2 = L^2$. Differentiating with respect to t and using the Power Rule for Functions:

$$2F \cdot \frac{dF}{dt} + 0 = 2L \cdot \frac{dL}{dt} \quad \Rightarrow \quad \frac{dF}{dt} = \frac{L}{F} \cdot \frac{dL}{dt}$$

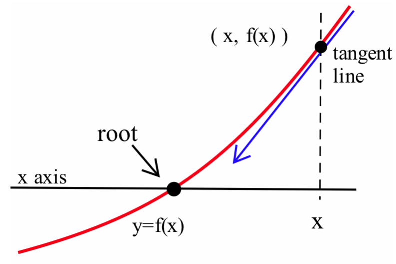
At a particular moment in time, $L = 60 \Rightarrow F^2 + 10^2 = 60^2 \Rightarrow F = \sqrt{3600 - 100} = \sqrt{3500} = 10\sqrt{35}$ so:

$$\left.\frac{dF}{dt}\right|_{L=60} = -30 \cdot \frac{60}{10\sqrt{35}} = -\frac{180}{\sqrt{35}} \approx -30.43\,\frac{\text{ft}}{\text{min}}$$



2.7 Newton's Method

Newton's method is a process that can find roots of functions whose graphs cross or just "kiss" the x-axis. Although this method is a bit harder to apply than the Bisection Algorithm, it often finds roots that the Bisection Algorithm misses, and it usually finds them faster.

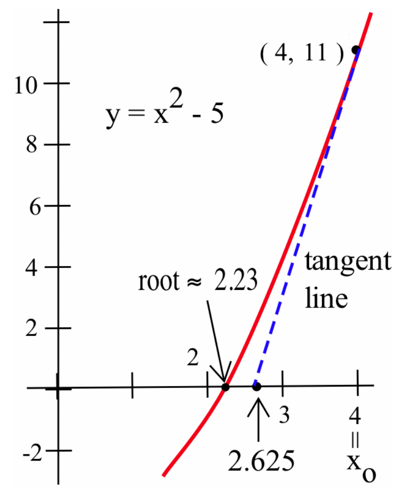


Off on a Tangent

The basic idea of Newton's Method is remarkably simple and graphical: at a point $(x, f(x))$ on the graph of f, the tangent line to the graph "points toward" a root of f, a place where the graph touches the x-axis.

To find a root of f, we just pick a starting value x_0, go to the point $(x_0, f(x_0))$ on the graph of f, build a tangent line there, and follow the tangent line to where it crosses the x-axis, say at x_1.

If x_1 is a root of f, we are done. If x_1 is not a root of f, then x_1 is usually closer to the root than x_0 was, and we can repeat the process, using x_1 as our new starting point. Newton's method is an **iterative** procedure—that is, the output from one application of the method becomes the starting point for the next application.

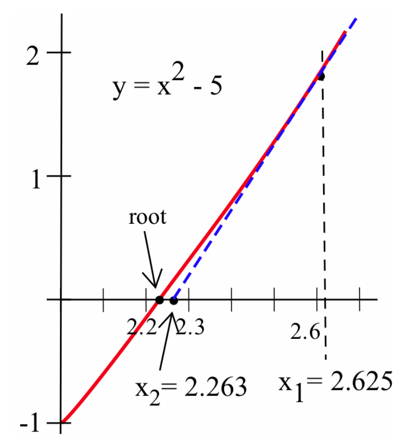


Let's begin with the function $f(x) = x^2 - 5$, whose roots we already know ($x = \pm\sqrt{5} \approx \pm 2.236067977$), to illustrate Newton's method. First, pick some value for x_0, say $x_0 = 4$, and move to the point $(x_0, f(x_0)) = (4, 11)$ on the graph of f. The tangent line to the graph of f at $(4, 11)$ "points to" a location on the x-axis that is closer to the root of f than the point we started with. We calculate this location on the x-axis by finding an equation of the line tangent to the graph of f at $(4, 11)$ and then finding where this line intersects the x-axis.

At $(4, 11)$, the line tangent to f has slope $f'(4) = 2(4) = 8$, so an equation of the tangent line is $y - 11 = 8(x - 4)$. Setting $y = 0$, we can find where this line crosses the x-axis:

$$0 - 11 = 8(x - 4) \Rightarrow x = 4 - \frac{11}{8} = \frac{21}{8} = 2.625$$

Call this new value x_1: The point $x_1 = 2.625$ is closer to the actual root $\sqrt{5}$, but it certainly does not equal the actual root. So we can use this new x-value, $x_1 = 2.625$, to repeat the procedure:

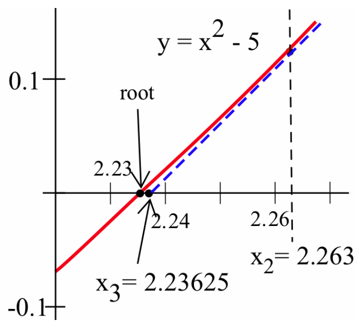


- move to the point $(x_1, f(x_1)) = (2.625, 1.890625)$
- find an equation of the tangent line at $(x_1, f(x_1))$:

$$y - 1.890625 = 5.25(x - 2.625)$$

- find x_2, the x-value where this new line intersects the x-axis:

$$y - 1.890625 = 5.25(x - 2.625) \Rightarrow 0 - 1.890625 = 5.25(x_2 - 2.625)$$
$$\Rightarrow x_2 = 2.264880952$$

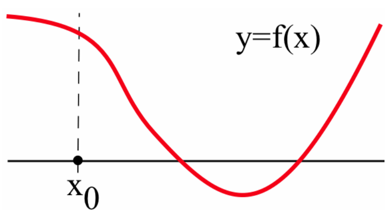


Repeating this process, each new estimate for the root of $f(x) = x^2 - 5$ becomes the starting point to calculate the next estimate. We get:

$x_0 = 4$	(0 correct digits)
$x_1 = 2.625$	(1 correct digit)
$x_2 = 2.262880952$	(2 correct digits)
$x_3 = 2.236251252$	(4 correct digits)
$x_4 = 2.236067985$	(8 correct digits)

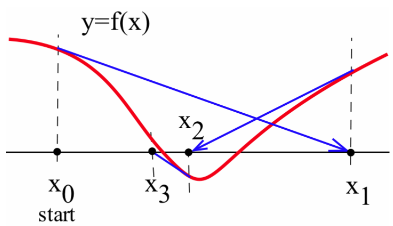
It only took 4 iterations to get an approximation within 0.000000008 of the exact value of $\sqrt{5}$. One more iteration gives an approximation x_5 that has 16 correct digits. If we start with $x_0 = -2$ (or any negative number), then the values of x_n approach $-\sqrt{5} \approx -2.23606$.

Practice 1. Find where the tangent line to $f(x) = x^3 + 3x - 1$ at $(1, 3)$ intersects the x-axis.



Practice 2. A starting point and a graph of f appear in the margin. Label the approximate locations of the next two points on the x-axis that will be found by Newton's method.

The Algorithm for Newton's Method



The process for Newton's Method, starting with x_0 and graphically finding the locations on the x-axis of x_1, x_2 and x_3.

Rather than deal with each particular function and starting point, let's find a pattern for a general function f.

For the starting point x_0, the slope of the tangent line at the point $(x_0, f(x_0))$ is $f'(x_0)$ so the equation of the tangent line is $y - f(x_0) = f'(x_0) \cdot (x - x_0)$. This line intersects the x-axis at a point $(x_1, 0)$, so:

$$0 - f(x_0) = f'(x_0) \cdot (x_1 - x_0) \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Starting with x_1 and repeating this process we get $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$, $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$ and so on. In general, starting with x_n, the line tangent to the graph of f at $(x_n, f(x_n))$ intersects the x-axis at $(x_{n+1}, 0)$ with $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, our new estimate for the root of f.

Algorithm for Newton's Method:

1. Pick a starting value x_0 (preferably close to a root of $f(x)$).
2. For each x_n, calculate a new estimate $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
3. Repeat step 2 until the estimates are "close enough" to a root or until the method "fails."

When we use Newton's method with $f(x) = x^2 - 5$, the function in our first example, we have $f'(x) = 2x$ so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n{}^2 - 5}{2x_n} = \frac{2x_n{}^2 - x_n{}^2 + 5}{2x_n}$$
$$= \frac{x_n{}^2 + 5}{2x_n} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$$

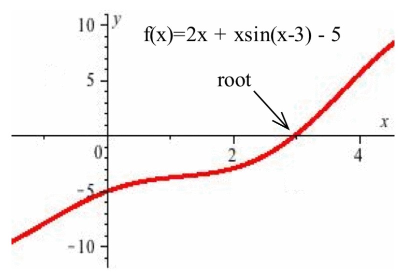
Problem 16 helps you show this pattern—called Heron's method—approximates the square root of any positive number: just replace 5 with the number whose square root you want to find.

The new approximation, x_{n+1}, is the average of the previous approximation, x_n, and 5 divided by the previous approximation, $\frac{5}{x_n}$.

Example 1. Use Newton's method to approximate the root(s) of $f(x) = 2x + x \cdot \sin(x+3) - 5$.

Solution. $f'(x) = 2 + x\cos(x+3) + \sin(x+3)$ so:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n + x_n \cdot \sin(x_n+3) - 5}{2 + x_n \cdot \cos(x_n+3) + \sin(x_n+3)}$$

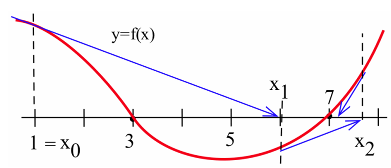


The graph of $f(x)$ (see margin) indicates only one root of f, which is near $x = 3$, so pick $x_0 = 3$. Then Newton's method yields the values $x_0 = 3$, $x_1 = \underline{2.964}84457$, $x_2 = \underline{2.96446}277$, $x_3 = \underline{2.96446273}$ (the underlined digits agree with the exact answer). ◀

If we had picked $x_0 = 4$ in the previous example, Newton's method would have required 4 iterations to get 9 digits of accuracy. For $x_0 = 5$, 7 iterations are needed to get 9 digits of accuracy. If we pick $x_0 = 5.1$, then the values of x_n are not close to the actual root after even 100 iterations: $x_{100} \approx -49.183$. Picking a "good" value for x_0 can result in values of x_n that get close to the root quickly. Picking a "poor" value for x_0 can result in x_n values that take many more iterations to get close to the root—or that don't approach the root at all.

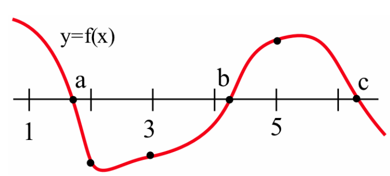
The graph of the function can help you pick a "good" x_0.

Practice 3. Put $x_0 = 3$ and use Newton's method to find the first two iterates, x_1 and x_2, for the function $f(x) = x^3 - 3x^2 + x - 1$.



Example 2. The function graphed in the margin has roots at $x = 3$ and $x = 7$. If we pick $x_0 = 1$ and apply Newton's method, which root do the iterates (the values of x_n) approach?

Solution. The iterates of $x_0 = 1$ are labeled in the margin graph. They are approaching the root at 7. ◀



Practice 4. For the function graphed in the margin, which root do the iterates of Newton's method approach if:

(a) $x_0 = 2$? (b) $x_0 = 3$? (c) $x_0 = 5$?

Iteration

We have been emphasizing the geometric nature of Newton's method, but Newton's method is also an example of **iterating a function**. If $N(x) = x - \dfrac{f(x)}{f'(x)}$, the "pattern" in the algorithm, then:

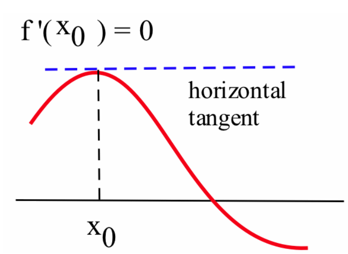
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = N(x_0)$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = N(x_1) = N(N(x_0)) = N \circ N(x_0)$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = N(x_2) = N(N(N(x_0))) = N \circ N \circ N(x_0)$$

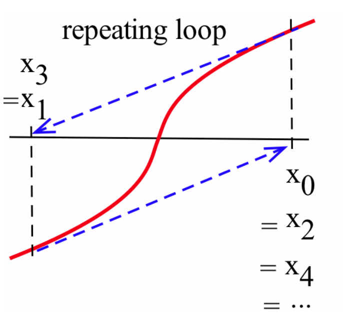
and, in general:

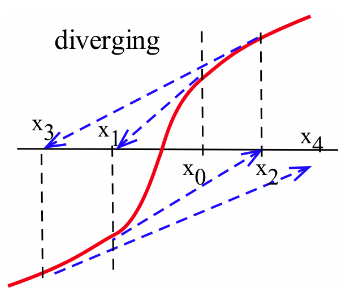
$$x_n = N(x_{n-1}) = n\text{th iteration of } N \text{ starting with } x_0$$

At each step, we use the output from N as the next input into N.

What Can Go Wrong?







When Newton's method works, it usually works very well and the values of x_n approach a root of f very quickly, often doubling the number of correct digits with each iteration. There are, however, several things that can go wrong.

An obvious problem with Newton's method is that $f'(x_n)$ can be 0. Then the algorithm tells us to divide by 0 and x_{n+1} is undefined. Geometrically, if $f'(x_n) = 0$, the tangent line to the graph of f at x_n is horizontal and does not intersect the x-axis at any point. If $f'(x_n) = 0$, just pick another starting value x_0 and begin again. In practice, a second or third choice of x_0 usually succeeds.

There are two other less obvious difficulties that are not as easy to overcome—the values of the iterates x_n may become locked into an infinitely repeating loop (see margin), or they may actually move farther away from a root (see lower margin figure).

Example 3. Put $x_0 = 1$ and use Newton's method to find the first two iterates, x_1 and x_2, for the function $f(x) = x^3 - 3x^2 + x - 1$.

Solution. This is the function from the previous Practice Problem, but with a different starting value for x_0: $f'(x) = 3x^2 - 6x + 1$ so,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-2}{-2} = 0$$
$$\text{and } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{1} = 1$$

which is the same as x_0, so $x_3 = x_1 = 0$ and $x_4 = x_2 = 1$. The values of x_n alternate between 1 and 0 and do not approach a root. ◀

Newton's method behaves badly at only a few starting points for this particular function — for most starting points, Newton's method converges to the root of this function. There are some functions, however, that defeat Newton's method for **almost every** starting point.

Practice 5. For $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ and $x_0 = 1$, verify that $x_1 = -2$, $x_2 = 4$ and $x_3 = -8$. Also try $x_0 = -3$ and verify that the same pattern holds: $x_{n+1} = -2x_n$. Graph f and explain why the Newton's method iterates get farther and farther away from the root at 0.

Newton's method is powerful and quick and very easy to program on a calculator or computer. It usually works so well that many people routinely use it as the first method they apply. If Newton's method fails for their particular function, they simply try some other method.

Chaotic Behavior and Newton's Method

An algorithm leads to **chaotic behavior** if two starting points that are close together generate iterates that are sometimes far apart and sometimes close together: $|a_0 - b_0|$ is small but $|a_n - b_n|$ is large for lots (infinitely many) of values of n and $|a_n - b_n|$ is small for lots of values of n. The iterates of the next simple algorithm exhibit chaotic behavior.

A Simple Chaotic Algorithm: Starting with any number between 0 and 1, double the number and keep the fractional part of the result: x_1 is the fractional part of $2x_0$, x_2 is the fractional part of $2x_1$, and in general, $x_{n+1} = 2x_n - \lfloor 2x_n \rfloor$.

If $x_0 = 0.33$, then the iterates of this algorithm are 0.66, 0.32 = fractional part of $2 \cdot 0.66$, 0.64, 0.28, 0.56, ... The iterates for two other starting values close to 0.33 are given below as well as the iterates of 0.470 and 0.471:

x_0	0.32	0.33	0.34	0.470	0.471
x_1	0.64	0.66	0.68	0.940	0.942
x_2	0.28	0.32	0.36	0.880	0.884
x_3	0.56	0.64	0.72	0.760	0.768
x_4	0.12	0.28	0.44	0.520	0.536
x_5	0.24	0.56	0.88	0.040	0.072
x_6	0.48	0.12	0.76	0.080	0.144
x_7	0.96	0.24	0.56	0.160	0.288
x_8	0.92	0.48	0.12	0.320	0.576
x_9	0.84	0.96	0.24	0.640	0.152

There are starting values as close together as we want whose iterates are far apart infinitely often.

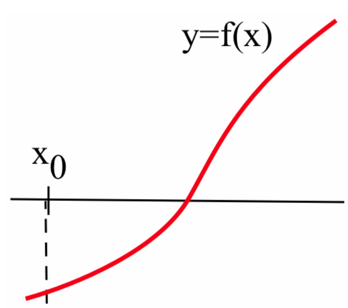
Many physical, biological and financial phenomena exhibit chaotic behavior. Atoms can start out within inches of each other and several weeks later be hundreds of miles apart. The idea that small initial differences can lead to dramatically diverse outcomes is sometimes called the "butterfly effect" from the title of a talk ("Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?") given by Edward Lorenz, one of the first people to investigate chaos. The "butterfly effect" has important implications about the possibility—or rather the impossibility—of accurate long-range weather forecasting. Chaotic behavior is also an important aspect of studying turbulent air and water flows, the incidence and spread of diseases, and even the fluctuating behavior of the stock market.

Newton's method often exhibits chaotic behavior and—because it is relatively easy to study—is often used as a model to investigate the properties of chaotic behavior. If we use Newton's method to approximate the roots of $f(x) = x^3 - x$ (with roots 0, +1 and −1), then starting points that are very close together can have iterates that converge to different roots. The iterates of 0.4472 and 0.4473 converge to the roots 0 and +1, respectively. The iterates of the median value 0.44725 converge to the root −1, and the iterates of another nearby point, $\frac{1}{\sqrt{5}} \approx 0.44721$, simply cycle between $-\frac{1}{\sqrt{5}}$ and $+\frac{1}{\sqrt{5}}$ and do not converge at all.

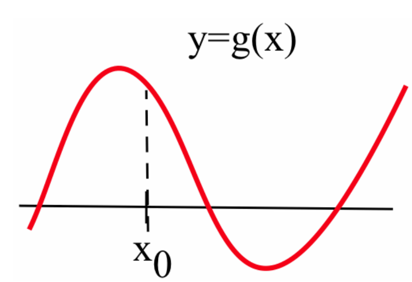
Practice 6. Find the first four Newton's method iterates of $x_0 = 0.997$ and $x_0 = 1.02$ for $f(x) = x^2 + 1$. Try two other starting values very close to 1 (but not equal to 1) and find their first four iterates. Use the graph of $f(x) = x^2 + 1$ to explain how starting points so close together can quickly have iterates so far apart.

2.7 Problems

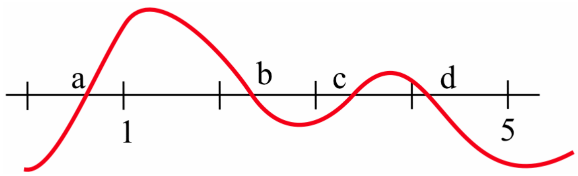
1. The graph of $y = f(x)$ appears below. Estimate the locations of x_1 and x_2 when you apply Newton's method with the given starting value x_0.



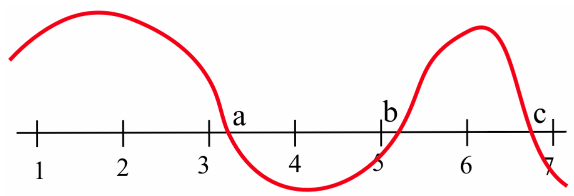
2. The graph of $y = g(x)$ appears below. Estimate the locations of x_1 and x_2 when you apply Newton's method starting value with the value x_0 shown in the graph.



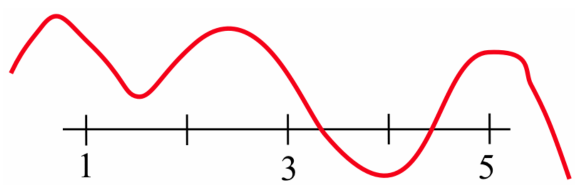
3. The function graphed below has several roots. Which root do the iterates of Newton's method converge to if we start with $x_0 = 1$? With $x_0 = 5$?



4. The function graphed below has several roots. Which root do the iterates of Newton's method converge to if we start with $x_0 = 2$? With $x_0 = 6$?



5. What happens to the iterates if we apply Newton's method to the function graphed below and start with $x_0 = 1$? With $x_0 = 5$?



6. What happens if we apply Newton's method to a function f and start with $x_0 =$ a root of f?
7. What happens if we apply Newton's method to a function f and start with $x_0 =$ a maximum of f?

In Problems 8–9, a function and a value for x_0 are given. Apply Newton's method to find x_1 and x_2.

8. $f(x) = x^3 + x - 1$ and $x_0 = 1$
9. $f(x) = x^4 - x^3 - 5$ and $x_0 = 2$

In Problems 10–11, use Newton's method to find a root, accurate to 2 decimal places, of the given functions using the given starting points.

10. $f(x) = x^3 - 7$ and $x_0 = 2$
11. $f(x) = x - \cos(x)$ and $x_0 = 0.7$

In Problems 12–15, use Newton's method to find all roots or solutions, accurate to 2 decimal places, of the given equation. It is helpful to examine a graph to determine a "good" starting value x_0.

12. $2 + x = e^x$
13. $\dfrac{x}{x+3} = x^2 - 2$
14. $x = \sin(x)$
15. $x = \sqrt[5]{3}$
16. Show that if you apply Newton's method to $f(x) = x^2 - A$ to approximate $\sqrt{A}$, then

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{A}{x_n}\right)$$

so the new estimate of the square root is the average of the previous estimate and A divided by the previous estimate. This method of approximating square roots is called Heron's method.

17. Use Newton's method to devise an algorithm for approximating the cube root of a number A.
18. Use Newton's method to devise an algorithm for approximating the n-th root of a number A.

Problems 19–22 involve chaotic behavior.

19. The iterates of numbers using the Simple Chaotic Algorithm have some interesting properties.
    (a) Verify that the iterates starting with $x_0 = 0$ are all equal to 0.
    (b) Verify that if $x_0 = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and, in general, $\frac{1}{2^n}$, then the n-th iterate of x_0 is 0 (and so are all iterates beyond the n-th iterate.)
20. When Newton's method is applied to the function $f(x) = x^2 + 1$, most starting values for x_0 lead to chaotic behavior for x_n. Find a value for x_0 so that the iterates alternate: $x_1 = -x_0$ and $x_2 = -x_1 = x_0$.

21. The function $f(x)$ defined as:

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2-2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

is called a "stretch and fold" function.

(a) Describe what f does to the points in the interval $[0, 1]$.

(b) Examine and describe the behavior of the iterates of $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$ and $\frac{2}{9}$.

(c) Examine and describe the behavior of the iterates of 0.10, 0.105 and 0.11.

(d) Do the iterates of f lead to chaotic behavior?

22. (a) After many iterations (50 is fine) what happens when you apply Newton's method starting with $x_0 = 0.5$ to:

i. $f(x) = 2x(1-x)$

ii. $f(x) = 3.3x(1-x)$

iii. $f(x) = 3.83x(1-x)$

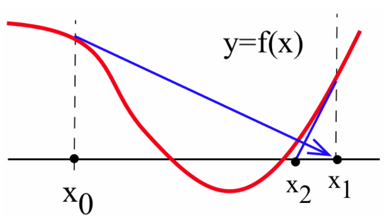
(b) What do you think happens to the iterates of $f(x) = 3.7x(1-x)$? What actually happens?

(c) Repeat parts (a)–(b) with some other starting values x_0 between 0 and 1 $(0 < x_0 < 1)$. Does the starting value seem to effect the eventual behavior of the iterates?

(The behavior of the iterates of f depends in a strange way on the numerical value of the leading coefficient. The behavior exhibited in part (b) is an example of "chaos.")

2.7 *Practice Answers*

1. $f'(x) = 3x^2 + 3$, so the slope of the tangent line at $(1, 3)$ is $f'(1) = 6$ and an equation of the tangent line is $y - 3 = 6(x - 1)$ or $y = 6x - 3$. The y-coordinate of a point on the x-axis is 0 so putting $y = 0$ in this equation: $0 = 6x - 3 \Rightarrow x = \frac{1}{2}$. The line tangent to the graph of $f(x) = x^3 + 3x + 1$ at the point $(1, 3)$ intersects the x-axis at the point $(\frac{1}{2}, 0)$.



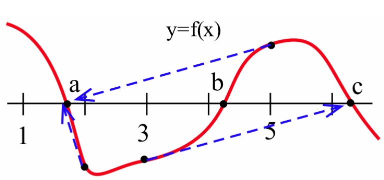
2. The approximate locations of x_1 and x_2 appear in the margin.

3. Using $f'(x) = 3x^2 + 3$ and $x_0 = 3$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{2}{10} = 2.8$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8 - \frac{f(2.8)}{f'(2.8)} = 2.8 - \frac{0.232}{7.72} \approx 2.769948187$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.769292663$$



4. The margin figure shows the first iteration of Newton's Method for $x_0 = 2$, 3 and 5: If $x_0 = 2$, the iterates approach the root at a; if $x_0 = 3$, they approach the root at c; and if $x_0 = 5$, they approach the root at a.

5. $f(x) = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. If $x_0 = 1$, then:

$$x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{\frac{1}{3}} = 1 - 3 = -2$$

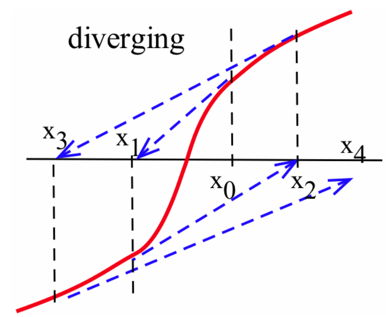
$$x_2 = -2 - \frac{f(-2)}{f'(-2)} = -2 - \frac{(-2)^{\frac{1}{3}}}{\frac{1}{3}(-2)^{-\frac{2}{3}}} = -2 - \frac{-2}{\frac{1}{3}} = 4$$

$$x_3 = 4 - \frac{f(4)}{f'(4)} = 4 - \frac{(4)^{\frac{1}{3}}}{\frac{1}{3}(4)^{-\frac{2}{3}}} = 4 - \frac{4}{\frac{1}{3}} = -8$$

and so on. If $x_0 = -3$, then:

$$x_1 = -3 - \frac{f(-3)}{f'(-3)} = -3 - \frac{(-3)^{\frac{1}{3}}}{\frac{1}{3}(-3)^{-\frac{2}{3}}} = -3 + 9 = 6$$

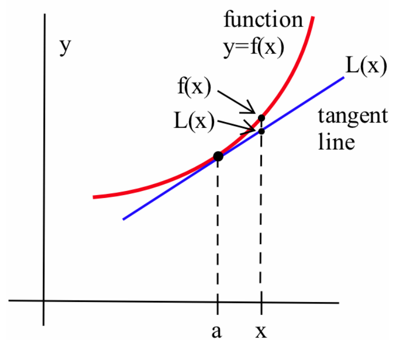
$$x_2 = 6 - \frac{f(6)}{f'(6)} = 6 - \frac{(6)^{\frac{1}{3}}}{\frac{1}{3}(6)^{-\frac{2}{3}}} = 6 - \frac{6}{\frac{1}{3}} = -12$$



The graph of $f(x) = \sqrt[3]{x}$ has a shape similar to the margin figure and the behavior of the iterates is similar to the pattern shown in that figure. Unless $x_0 = 0$ (the only root of f) the iterates alternate in sign and double in magnitude with each iteration: they get progressively farther from the root with each iteration.

6. If $x_0 = 0.997$, then $x_1 \approx -0.003$, $x_2 \approx 166.4$, $x_3 \approx 83.2$, $x_4 \approx 41.6$. If $x_0 = 1.02$, then $x_1 \approx 0.0198$, $x_2 \approx -25.2376$, $x_3 \approx -12.6$ and $x_4 \approx -6.26$.

2.8 *Linear Approximation and Differentials*



Newton's method used tangent lines to "point toward" a root of a function. In this section we examine and use another geometric characteristic of tangent lines:

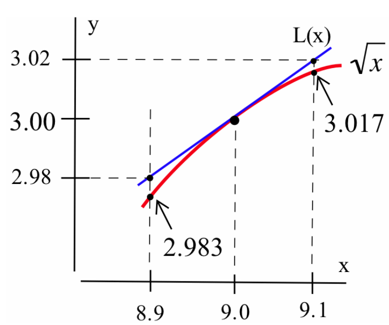
If	f is differentiable at a, c is close to a
	and $y = L(x)$ is the line tangent to $f(x)$ at $x = a$
then	$L(c)$ is close to $f(c)$.

We can use this idea to approximate the values of some commonly used functions and to predict the "error" or uncertainty in a computation if we know the "error" or uncertainty in our original data. At the end of this section, we will define a related concept called the **differential** of a function.

Linear Approximation

Because this section uses tangent lines extensively, it is worthwhile to recall how we find the equation of the line tangent to $f(x)$ where $x = a$: the tangent line goes through the point $(a, f(a))$ and has slope $f'(a)$ so, using the point-slope form $y - y_0 = m(x - x_0)$ for linear equations, we have $y - f(a) = f'(a) \cdot (x - a) \Rightarrow y = f(a) + f'(a) \cdot (x - a)$.

If	f is differentiable at $x = a$
then	an equation of the line L tangent to f at $x = a$ is:
	$L(x) = f(a) + f'(a) \cdot (x - a)$



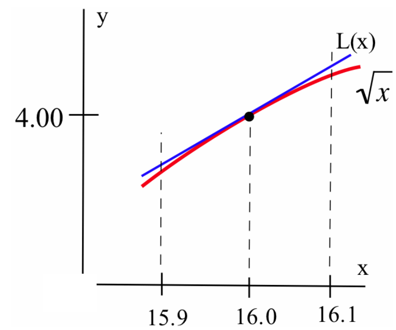
Example 1. Find a formula for $L(x)$, the linear function tangent to the graph of $f(x) = \sqrt{x}$ at the point $(9, 3)$. Evaluate $L(9.1)$ and $L(8.88)$ to approximate $\sqrt{9.1}$ and $\sqrt{8.88}$.

Solution. $f(x) = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ so $f(9) = 3$ and $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$. Thus:

$$L(x) = f(9) + f'(9) \cdot (x - 9) = 3 + \frac{1}{6}(x - 9)$$

If x is close to 9, then the value of $L(x)$ should be a good approximation of the value of x. The number 9.1 is close to 9 so $\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.016666$. Similarly, $\sqrt{8.88} = f(8.88) \approx L(8.88) = 3 + \frac{1}{6}(8.88 - 9) = 2.98$. In fact, $\sqrt{9.1} \approx 3.016621$, so our estimate using $L(9.1)$ is within 0.000045 of the exact answer; $\sqrt{8.88} \approx 2.979933$ (accurate to 6 decimal places) and our estimate is within 0.00007 of the exact answer. ◀

In each case in the previous example, we got a good estimate of a square root with very little work. The graph in the margin indicates the graph of the tangent line $y = L(x)$ lies slightly above the graph of $y = f(x)$; consequently (as we observed), each estimate is slightly larger than the exact value.



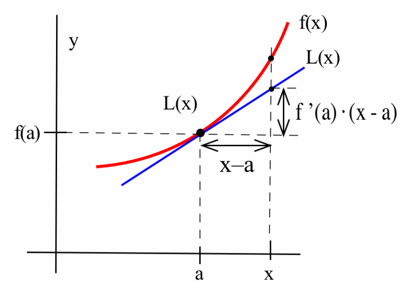
Practice 1. Find a formula for $L(x)$, the linear function tangent to the graph of $f(x) = \sqrt{x}$ at the point $(16, 4)$. Evaluate $L(16.1)$ and $L(15.92)$ to approximate $\sqrt{16.1}$ and $\sqrt{15.92}$. Are your approximations using L larger or smaller than the exact values of the square roots?

Practice 2. Find a formula for $L(x)$, the linear function tangent to the graph of $f(x) = x^3$ at the point $(1, 1)$ and use $L(x)$ to approximate $(1.02)^3$ and $(0.97)^3$. Do you think your approximations using L are larger or smaller than the exact values?

The process we have used to approximate square roots and cubics can be used to approximate values of any differentiable function, and the main result about the linear approximation follows from the two statements in the boxes. Putting these two statements together, we have the process for Linear Approximation.

Linear Approximation Process:

If	f is differentiable at a and $L(x) = f(a) + f'(a) \cdot (x - a)$
then	(geometrically) the graph of $L(x)$ is close to the graph of $f(x)$ when x is close to a
and	(algebraically) the values of the $L(x)$ approximate the values of $f(x)$ when x is close to a: $f(x) \approx L(x) = f(a) + f'(a) \cdot (x - a)$



Sometimes we replace "$x - a$" with "Δx" in the last equation, and the statement becomes $f(x) \approx f(a) + f'(a) \cdot \Delta x$.

Example 2. Use the linear approximation process to approximate the value of $e^{0.1}$.

Solution. $f(x) = e^x \Rightarrow f'(x) = e^x$ so we need to pick a value a near $x = 0.1$ for which we know the exact value of $f(a) = e^a$ and $f'(a) = e^a$: $a = 0$ is an obvious choice. Then:

$$\begin{aligned} e^{0.1} &= f(0.1) \approx L(0.1) = f(0) + f'(0) \cdot (0.1 - 0) \\ &= e^0 + e^0 \cdot (0.1) = 1 + 1 \cdot (0.1) = 1.1 \end{aligned}$$

You can use your calculator to verify that this approximation is within 0.0052 of the exact value of $e^{0.1}$. ◀

Practice 3. Approximate the value of $(1.06)^4$, the amount $1 becomes after 4 years in a bank account paying 6% interest compounded annually. (Take $f(x) = x^4$ and $a = 1$.)

Practice 4. Use the linear approximation process and the values in the table below to estimate the value of f when $x = 1.1$, 1.23 and 1.38.

x	$f(x)$	$f'(x)$
1.0	0.7854	0.5
1.2	0.8761	0.4098
1.4	0.9505	0.3378

We can approximate **functions** as well as numbers (specific values of those functions).

Example 3. Find a linear approximation formula $L(x)$ for $\sqrt{1+x}$ when x is small. Use your result to approximate $\sqrt{1.1}$ and $\sqrt{0.96}$.

Solution. $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} = \frac{1}{2\sqrt{1+x}}$, so because "$x$ is small," we know that x is close to 0 and we can pick $a = 0$. Then $f(a) = f(0) = 1$ and $f'(a) = f'(0) = \frac{1}{2}$ so

$$\sqrt{1+x} \approx L(x) = f(0) + f'(0) \cdot (x-0) = 1 + \frac{1}{2}x = 1 + \frac{x}{2}$$

Taking $x = 0.1$, $\sqrt{1.1} = \sqrt{1+0.1} \approx 1 + \frac{0.1}{2} = 1.05$; taking $x = -0.04$, $\sqrt{0.96} = \sqrt{1+(-.04)} \approx 1 + \frac{-0.04}{2} = 0.98$. Use your calculator to determine by how much each estimate differs from the true value. ◀

Applications of Linear Approximation to Measurement "Error"

Most scientific experiments use instruments to take measurements, but these instruments are not perfect, and the measurements we get from them are only accurate up to a certain level of precision. If we know this level of accuracy of our instruments and measurements, we can use the idea of linear approximation to estimate the level of accuracy of results we calculate from our measurements.

If we measure the side x of a square to be 8 inches, then we would of course calculate its area to be $8^2 = 64$ square inches. Suppose, as would reasonable with a real measurement, that our measuring instrument could only measure or be read to the nearest 0.05 inches. Then our measurement of 8 inches would really mean some number between $8 - 0.05 = 7.95$ inches and $8 + 0.05 = 8.05$ inches, so the true area of the square would be between $7.95^2 = 63.2025$ and $8.05^2 = 64.8025$ square inches. Our possible "error" or "uncertainty," because of the limitations of the instrument, could be as much as $64.8025 - 64 = 0.8025$ square inches, so we could report the area of the square to be 64 ± 0.8025

square inches. We can also use the linear approximation method to estimate the "error" or uncertainty of the area.

For a function as simple as the area of a square, this linear approximation method really isn't needed, but it serves as a useful and easily understood illustration of the technique.

For a square with side x, the area is $A(x) = x^2$ and $A'(x) = 2x$. If Δx represents the "error" or uncertainty of our measurement of the side then, using the linear approximation technique for $A(x)$, $A(x) \approx A(a) + A'(a) \cdot \Delta x$ so the uncertainty of our calculated area is $A(x) - A(a) \approx A'(a) \cdot \Delta x$. In this example, $a = 8$ inches and $\Delta x = 0.05$ inches, so $A(8.05) \approx A(8) + A'(8) \cdot (0.05) = 64 + 2(8) \cdot (0.05) = 64.8$ square inches, and the uncertainty in our calculated area is approximately $A(8+0.05) - A(8) \approx A'(8) \cdot \Delta x = 2(8 \text{ inches})(0.05 \text{ inches}) =$ 0.8 square inches. (Compare this approximation of the biggest possible error with the exact answer of 0.8025 square inches computed previously.) This process can be summarized as:

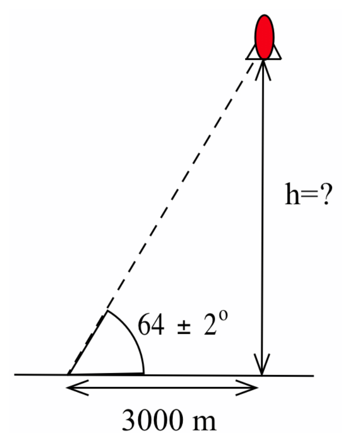
Linear Approximation Error:

If the value of the x-variable is measured to be $x = a$ with a maximum "error" of Δx units

then Δf, the "error" in estimating $f(x)$, is:

$$\Delta f = f(x) - f(a) \approx f'(a) \cdot \Delta x.$$

Practice 5. If we measure the side of a cube to be 4 cm with an uncertainty of 0.1 cm, what is the volume of the cube and the uncertainty of our calculation of the volume? (Use linear approximation.)



Example 4. We are using a tracking telescope to follow a small rocket. Suppose we are 3,000 meters from the launch point of the rocket, and, 2 seconds after the launch we measure the angle of the inclination of the rocket to be 64° with a possible "error" of 2°. How high is the rocket and what is the possible error in this calculated height?

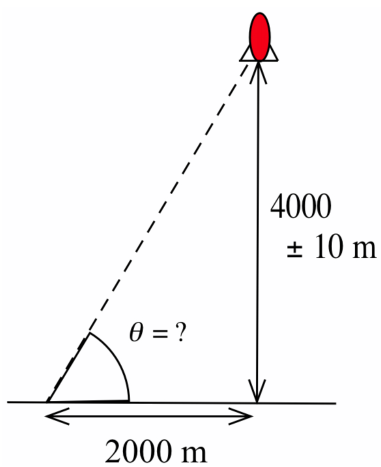
Solution. Our measured angle is $x = 1.1170$ radians with $\Delta x = 0.0349$ radians (all trigonometric work should be in radians), and the height of the rocket at an angle x is $f(x) = 3000\tan(x)$ so $f(1.1170) \approx 6151$ m. Our uncertainty in the height is $\Delta f \approx f'(x) \cdot \Delta x \approx 3000 \cdot \sec^2(x) \cdot \Delta x =$ $3000\sec^2(1.1170) \cdot 0.0349 \approx 545$ m. If our measured angle of 64° can be in error by as much as 2°, then our calculated height of 6,151 m can be in error by as much as 545 m. The height is 6151 ± 545 meters. ◀

Practice 6. Suppose we measured the angle of inclination in the previous Example to be $43° \pm 1°$. Estimate the height of the rocket in the form "height ± error."

In some scientific and engineering applications, the calculated **result** must be within some given specification. You might need to determine how accurate the initial measurements must be in order to guarantee the final calculation is within that specification. Added precision usually costs time and money, so it is important to choose a measuring instrument good enough for the job but which is not too expensive.

Example 5. Your company produces ball bearings (small metal spheres) with a volume of 10 cm^3 and the volume must be accurate to within 0.1 cm^3. What radius should the bearings have—and what error can you tolerate in the radius measurement to meet the accuracy specification for the volume?

Solution. We want $V = 10$ and we know that the volume of a sphere is $V = \frac{4}{3}\pi r^3$, so solve $10 = \frac{4}{3}\pi r^3$ for r to get $r = 1.3365$ cm. $V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = 4\pi r^2$ so $\Delta V \approx V'(r) \cdot \Delta r$. In this case we know that $\Delta V = 0.1 \text{ cm}^3$ and we have calculated $r = 1.3365$ cm, so $0.1 \text{ cm}^3 = V'(1.3365 \text{ cm}) \cdot \Delta r = (22.45 \text{ cm}^2) \cdot \Delta r$. Solving for Δr, we get $\Delta r \approx 0.0045$ cm. To meet the specification for allowable error in volume, we must allow the radius to vary no more than 0.0045 cm. If we instead measure the diameter of the sphere, then we want the diameter to be $d = 2r = 2(1.3365 \pm 0.0045) = 2.673 \pm 0.009$ cm. ◀



Practice 7. You want to determine the height of a rocket to within 10 meters when it is 4,000 meters high (see margin figure). How accurate must your angle of measurement be? (Do your calculations in radians.)

Relative Error and Percentage Error

The "error" we've been examining is called the **absolute error** to distinguish it from two other terms, the **relative error** and the **percentage error**, which compare the absolute error with the magnitude of the number being measured. An "error" of 6 inches in measuring the Earth's circumference would be extremely small, but a 6-inch error in measuring your head for a hat would result in a very bad fit.

Definitions:

The **Relative** Error of f is $\dfrac{\text{error of } f}{\text{value of } f} = \dfrac{\Delta f}{f}$

The **Percentage** Error of f is $\dfrac{\Delta f}{f} \cdot 100\%$.

Example 6. If the relative error in the calculation of the area of a circle must be less than 0.4, then what relative error can we tolerate in the measurement of the radius?

Solution. $A(r) = \pi r^2 \Rightarrow A'(r) = 2\pi r$ and $\Delta A \approx A'(r) \cdot \Delta r = 2\pi r \Delta r$. The Relative Error of A is:

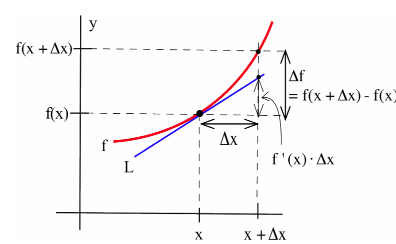
$$\frac{\Delta A}{A} \approx \frac{2\pi r \Delta r}{\pi r^2} = 2\frac{\Delta r}{r}$$

We can guarantee that the Relative Error of A, $\frac{\Delta A}{A}$, is less than 0.4 if the Relative Error of r, $\frac{\Delta r}{r} = \frac{1}{2}\frac{\Delta A}{A}$, is less than $\frac{1}{2}(0.4) = 0.2$. ◀

Practice 8. If you can measure the side of a cube with a percentage error less than 3%, then what will the percentage error for your calculation of the surface area of the cube be?

The Differential of f

As shown in the margin, the change in value of the function f near the point $(x, f(x))$ is $\Delta f = f(x + \Delta x) - f(x)$ and the change along the tangent line is $f'(x) \cdot \Delta x$. If Δx is small, then we have used the approximation that $\Delta f \approx f'(x) \cdot \Delta x$. This leads to the definition of a new quantity, df, called the **differential** of f.



Definition:

The **differential** of f is $df = f'(x) \cdot dx$ where dx is any real number.

The differential of f represents the change in f, as x changes from x to $x + dx$, along the tangent line to the graph of f at the point $(x, f(x))$. If we take dx to be the number Δx, then the differential is an approximation of Δf: $\Delta f \approx f'(x) \cdot \Delta x = f'(x) \cdot dx = df$.

Example 7. Determine the differential for the functions $f(x) = x^3 - 7x$, $g(x) = \sin(x)$ and $h(r) = \pi r^2$.

While we will do very little with differentials for a while, we will use them extensively in integral calculus.

Solution. $df = f'(x) \cdot dx = (3x^2 - 7)\,dx$, $dg = g'(x) \cdot dx = \cos(x)\,dx$, and $dh = h'(r)\,dr = 2\pi r\,dr$. ◀

Practice 9. Determine the differentials of $f(x) = \ln(x)$, $u = \sqrt{1 - 3x}$ and $r = 3\cos(\theta)$.

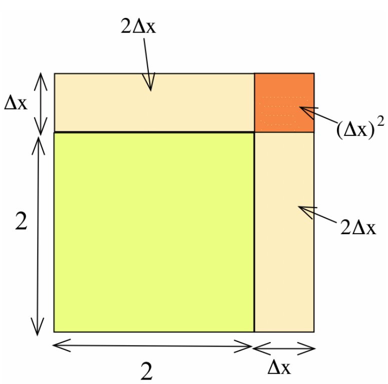
The Linear Approximation "Error" $|f(x) - L(x)|$

An approximation is most valuable if we also have have some measure of the size of the "error," the distance between the approximate value and the value being approximated. Typically, we will not know the exact value of the error (why not?), but it is useful to know an upper bound for the error. For example, if one scale gives the weight of a gold pendant as 10.64 grams with an error less than 0.3 grams (10.64 ± 0.3 grams) and another scale gives the weight of the same pendant as 10.53 grams with an error less than 0.02 grams (10.53 ± 0.02 grams), then we can have more faith in the second approximate weight because of the smaller "error" guarantee. Before finding a guarantee on the size of the error of the linear approximation process, we will check how well the linear approximation process approximates values of some functions we can compute exactly. Then we will prove one bound on the possible error and state a somewhat stronger bound.

Example 8. Given the function $f(x) = x^2$, evaluate the expressions $f(2+\Delta x)$, $L(2+\Delta x)$ and $|f(2+\Delta x) - L(2+\Delta x)|$ for $\Delta x = 0.1$, 0.05, 0.01, 0.001 and for a general value of Δx.

Solution. $f(2+\Delta x) = (2+\Delta x)^2 = 2^2 + 4\Delta x + (\Delta x)^2$ and $L(2+\Delta x) = f(2) + f'(2) \cdot \Delta x = 2^2 + 4 \cdot \Delta x$. Then:

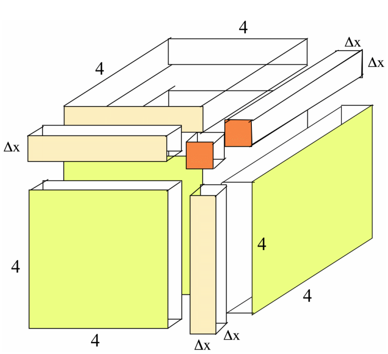
Δx	$f(2+\Delta x)$	$L(2+\Delta x)$	$\|f(2+\Delta x) - L(2+\Delta x)\|$
0.1	4.41	4.4	0.01
0.05	4.2025	4.2	0.0025
0.01	4.0401	4.04	0.0001
0.001	4.004001	4.004	0.000001



Cutting the value of Δx in half makes the error one fourth as large. Cutting Δx to $\frac{1}{10}$ as large makes the error $\frac{1}{100}$ as large. In general:

$$\begin{aligned} |f(2+\Delta x) - L(2+\Delta x)| &= \left| \left(2^2 + 4 \cdot \Delta x + (\Delta x)^2\right) - \left(2^2 + 4 \cdot \Delta x\right) \right| \\ &= (\Delta x)^2 \end{aligned}$$

This function and error also have a nice geometric interpretation (see margin): $f(x) = x^2$ is the area of a square of side x so $f(2+\Delta x)$ is the area of a square of side $2+\Delta x$, and that area is the sum of the pieces with areas 2^2, $2 \cdot \Delta x$, $2 \cdot \Delta x$ and $(\Delta x)^2$. The linear approximation $L(2+\Delta x) = 2^2 + 4 \cdot \Delta x$ to the area of the square includes the three largest pieces, 2^2, $2 \cdot \Delta x$ and $2 \cdot \Delta x$, but omits the small square with area $(\Delta x)^2$ so the approximation is in error by the amount $(\Delta x)^2$. ◀



Practice 10. Given $f(x) = x^3$, evaluate $f(4+\Delta x)$, $L(4+\Delta x)$ and $|f(4+\Delta x) - L(4+\Delta x)|$ for $\Delta x = 0.1, 0.05, 0.01, 0.001$ and for a general value of Δx. Use the margin figure to give a geometric interpretation of $f(4+\Delta x)$, $L(4+\Delta x)$ and $|f(4+\Delta x) - L(4+\Delta x)|$.

In the previous Example and previous Practice problem, the error $|f(a+\Delta x) - L(a+\Delta x)|$ was very small, proportional to $(\Delta x)^2$, when Δx was small. In general, this error approaches 0 as $\Delta x \to 0$.

Theorem:

If $f(x)$ is differentiable at a
and $L(a+\Delta x) = f(a) + f'(a)\cdot\Delta x$
then $\lim_{\Delta x\to 0} |f(a+\Delta x) - L(a+\Delta x)| = 0$
and $\lim_{\Delta x\to 0} \dfrac{|f(a+\Delta x) - L(a+\Delta x)|}{\Delta x} = 0.$

Proof. First rewrite the quantity inside the absolute value as:

$$\begin{aligned} f(a+\Delta x) - L(a+\Delta x) &= f(a+\Delta x) - f(a) - f'(a)\cdot\Delta x \\ &= \left[\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right]\cdot\Delta x \end{aligned}$$

But f is differentiable at $x = a$ so $\lim_{\Delta x\to 0} \dfrac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a)$, which we can rewrite as $\lim_{\Delta x\to 0} \left[\dfrac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right] = 0$. Thus:

$$\lim_{\Delta x\to 0} [f(a+\Delta x) - L(a+\Delta x)] = \lim_{\Delta x\to 0} \left[\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right]\cdot \lim_{\Delta x\to 0} \Delta x = 0\cdot 0 = 0$$

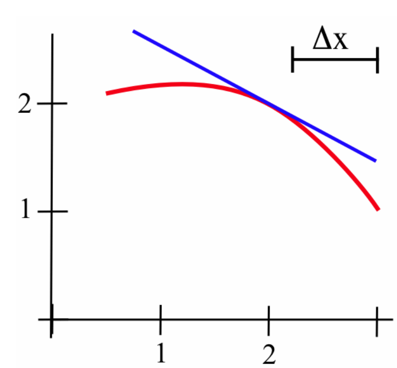
Not only does the difference $f(a+\Delta x) - L(a+\Delta x)$ approach 0, but this difference approaches 0 so fast that we can divide it by Δx, another quantity approaching 0, and the quotient still approaches 0. □

In the next chapter we will be able to prove that the error of the linear approximation process is in fact proportional to $(\Delta x)^2$. For now, we just state the result.

Theorem:

If f is differentiable at a
and $|f''(x)| \le M$ for all x between a and $a+\Delta x$
then $|\text{"error"}| = |f(a+\Delta x) - L(a+\Delta x)| \le \frac{1}{2}M\cdot(\Delta x)^2.$

2.8 Problems



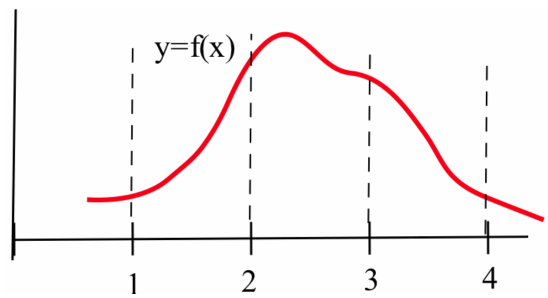
1. The figure in the margin shows the tangent line to a function g at the point $(2,2)$ and a line segment Δx units long.

   (a) On the figure, label the locations of

      i. $2+\Delta x$ on the x-axis
      ii. the point $(2+\Delta x, g(2+\Delta x))$
      iii. the point $(2+\Delta x, g(2) + g'(2)\cdot\Delta x)$

   (b) How large is the "error," $(g(2) + g'(2)\cdot\Delta x) - (g(2+\Delta x))$?

2. In the figure below, is the linear approximation $L(a + \Delta x)$ larger or smaller than the value of $f(a + \Delta x)$ when:
   (a) $a = 1$ and $\Delta x = 0.2$?
   (b) $a = 2$ and $\Delta x = -0.1$?
   (c) $a = 3$ and $\Delta x = 0.1$?
   (d) $a = 4$ and $\Delta x = 0.2$?
   (e) $a = 4$ and $\Delta x = -0.2$?

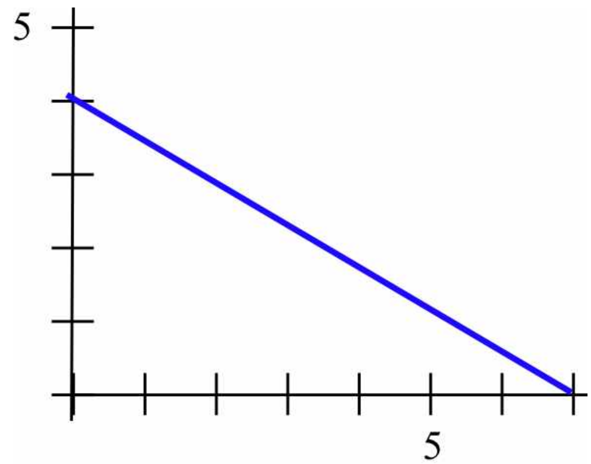


In Problems 3–4, find a formula for the linear function $L(x)$ tangent to the given function f at the given point $(a, f(a))$. Use the value $L(a + \Delta x)$ to approximate the value of $f(a + \Delta x)$.

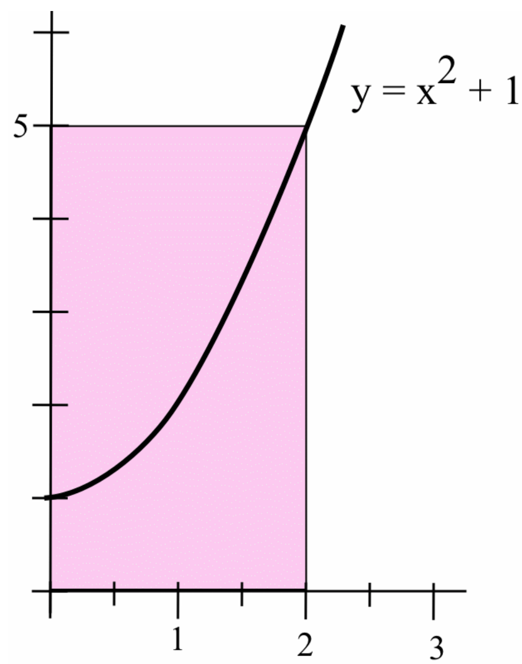
3. (a) $f(x) = \sqrt{x}$, $a = 4$, $\Delta x = 0.2$
   (b) $f(x) = \sqrt{x}$, $a = 81$, $\Delta x = -1$
   (c) $f(x) = \sin(x)$, $a = 0$, $\Delta x = 0.3$
4. (a) $f(x) = \ln(x)$, $a = 1$, $\Delta x = 0.3$
   (b) $f(x) = e^x$, $a = 0$, $\Delta x = 0.1$
   (c) $f(x) = x^5$, $a = 1$, $\Delta x = 0.03$
5. Show that $(1 + x)^n \approx 1 + nx$ if x is "close to" 0. (Suggestion: Put $f(x) = (1 + x)^n$ and $a = 0$ and then replace Δx with x.)

In 6–7, use the linear approximation process to obtain each formula for x "close to" 0.

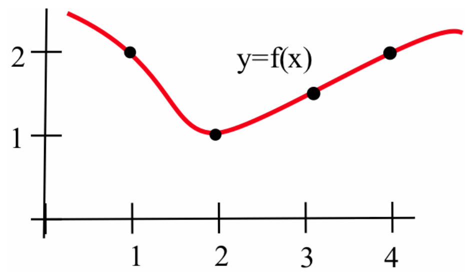
6. (a) $(1 - x)^n \approx 1 - nx$
   (b) $\sin(x) \approx x$
   (c) $e^x \approx 1 + x$
7. (a) $\ln(1 + x) \approx x$
   (b) $\cos(x) \approx 1$
   (c) $\tan(x) \approx x$
   (d) $\sin\left(\frac{\pi}{2} + x\right) \approx 1$
8. The height of a triangle is exactly 4 inches, and the base is measured to be 7±0.5 inches (see figure below). Shade a part of the figure that represents the "error" in the calculation of the area of the triangle.



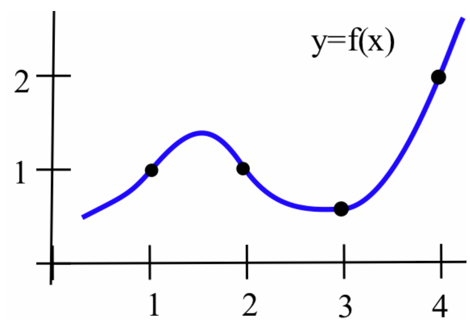
9. A rectangle has one side on the x-axis, one side on the y-axis and a corner on the graph of $y = x^2 + 1$ (see figure below).
   (a) Use Linear Approximation of the area formula to estimate the increase in the area of the rectangle if the base grows from 2 to 2.3 inches.
   (b) Calculate exactly the increase in the area of the rectangle as the base grows from 2 to 2.3 inches.



10. You know that you can measure the diameter of a circle to within 0.3 cm of the exact value.
   (a) How large is the "error" in the calculated area of a circle with a measured diameter of 7.4 cm?
   (b) How large is the "error" with a measured diameter of 13.6 cm?
   (c) How large is the percentage error in the calculated area with a measured diameter of d?
11. You are minting gold coins that must have a volume of 47.3±0.1 cm^3. If you can manufacture the coins to be exactly 2 cm high, how much variation can you allow for the radius?
12. If F is the fraction of carbon-14 remaining in a plant sample Y years after it died, then $Y = 5700 \ln(0.5) \cdot \ln(F)$.
   (a) Estimate the age of a plant sample in which 83±2% (0.83 ± 0.02) of the carbon-14 remains.
   (b) Estimate the age of a plant sample in which 13±2% (0.13 ± 0.02) of the carbon-14 remains.
13. Your company is making dice (cubes) and specifications require that their volume be 87±2 cm^3. How long should each side be and how much variation can be allowed?
14. If the specifications require a cube with a surface area of 43±0.2 cm^2, how long should each side be and how much variation can be allowed in order to meet the specifications?
15. The period P, in seconds, for a pendulum to make one complete swing and return to the release point is $P = 2\pi\sqrt{\frac{L}{g}}$ where L is the length of the pendulum in feet and g is 32 feet/sec^2.
   (a) If $L = 2$ feet, what is the period?
   (b) If $P = 1$ second, how long is the pendulum?
   (c) Estimate the change in P if L increases from 2 feet to 2.1 feet.
   (d) The length of a 24-foot pendulum is increasing 2 inches per hour. Is the period getting longer or shorter? How fast is the period changing?
16. A ball thrown at an angle θ (with the horizontal) with an initial velocity v will land $\frac{v^2}{g} \cdot \sin(2\theta)$ feet from the thrower.
   (a) How far away will the ball land if $\theta = \frac{\pi}{4}$ and $v = 80$ feet/second?
   (b) Which will result in a greater change in the distance: a 5% error in the angle θ or a 5% error in the initial velocity v?
17. For the function graphed below, estimate the value of df when
   (a) $x = 2$ and $dx = 1$
   (b) $x = 4$ and $dx = -1$
   (c) $x = 3$ and $dx = 2$



18. For the function graphed below, estimate the value of df when
   (a) $x = 1$ and $dx = 2$
   (b) $x = 2$ and $dx = -1$
   (c) $x = 3$ and $dx = 1$



19. Calculate the differentials df for the following functions:
   (a) $f(x) = x^2 - 3x$
   (b) $f(x) = e^x$
   (c) $f(x) = \sin(5x)$
   (d) $f(x) = x^3 + 2x$ with $x = 1$ and $dx = 0.2$
   (e) $f(x) = \ln(x)$ with $x = e$ and $dx = -0.1$
   (f) $f(x) = \sqrt{2x+5}$ with $x = 22$ and $dx = 3$.

2.8 Practice Answers

1. $f(x) = x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$. At the point $(16, 4)$ on the graph of f, the slope of the tangent line is $f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{8}$. An equation of the tangent line is $y - 4 = \frac{1}{8}(x - 16)$ or $y = \frac{1}{8}x + 2$: $L(x) = \frac{1}{8}x + 2$. So:

$$\sqrt{16.1} \approx L(16.1) = \frac{1}{8}(16.1) + 2 = 4.0125$$
$$\sqrt{15.92} \approx L(15.92) = \frac{1}{8}(15.92) + 2 = 3.99$$

2. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$. At $(1, 1)$, the slope of the tangent line is $f'(1) = 3$. An equation of the tangent line is $y - 1 = 3(x - 1)$ or $y = 3x - 2$: $L(x) = 3x - 2$. So:

$$(1.02)^3 \approx L(1.02) = 3(1.02) - 2 = 1.06$$
$$(0.97)^3 \approx L(0.97) = 3(0.97) - 2 = 0.91$$

3. $f(x) = x^4 \Rightarrow f'(x) = 4x^3$. Taking $a = 1$ and $\Delta x = 0.06$:

$$\begin{aligned}(1.06)^4 = f(1.06) &\approx L(1.06) = f(1) + f'(1) \cdot (0.06) \\ &= 1^4 + 4(1^3)(0.06) = 1.24\end{aligned}$$

4. Using values given in the table:

$$\begin{aligned}f(1.1) &\approx f(1) + f'(1) \cdot (0.1) \\ &= 0.7854 + (0.5)(0.1) = 0.8354 \\ f(1.23) &\approx f(1.2) + f'(1.2) \cdot (0.03) \\ &= 0.8761 + (0.4098)(0.03) = 0.888394 \\ f(1.38) &\approx f(1.4) + f'(1.4) \cdot (-0.02) \\ &= 0.9505 + (0.3378)(-0.02) = 0.943744\end{aligned}$$

5. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$ so $f(4) = 4^3 = 64$ cm^3 and the "error" is:

$$\Delta f \approx f'(x) \cdot \Delta x = 3x^2 \cdot \Delta x$$

When $x = 4$ and $\Delta x = 0.1$, $\Delta f \approx 3(4)^2(0.1) = 4.8$ cm^3.

6. $43^\circ \pm 1^\circ$ is equivalent to 0.75049 ± 0.01745 radians, so with $f(x) = 3000\tan(x)$ we have $f(0.75049) = 3000\tan(0.75049) \approx 2797.5$ m and $f'(x) = 3000\sec^2(x)$. So:

$$\begin{aligned}\Delta f &\approx f'(x) \cdot \Delta x = 3000\sec^2(x) \cdot \Delta x \\ &= 3000\sec^2(0.75049) \cdot (0.01745) = 97.9 \text{ m}\end{aligned}$$

The height of the rocket is 2797.5 ± 97.9 m.

7. $f(\theta) = 2000\tan(\theta) \Rightarrow f'(\theta) = 2000\sec^2(\theta)$ and we know $f(\theta) = 4000$, so:

$$4000 = 2000\tan(\theta) \Rightarrow \tan(\theta) = 2 \Rightarrow \theta \approx 1.10715 \text{ (radians)}$$

and thus $f'(1.10715) = 2000\sec^2(1.10715) \approx 10000$. Finally, the "error" is given by $\Delta f \approx f'(\theta) \cdot \Delta\theta$ so:

$$10 \approx 10000 \cdot \Delta\theta \;\Rightarrow\; \Delta\theta \approx \frac{10}{10000} = 0.001 \text{ (radians)} \;\approx 0.057^\circ$$

8. $A(r) = 6r^2 \Rightarrow A'(r) = 12r \Rightarrow \Delta A \approx A'(r) \cdot \Delta r = 12r \cdot \Delta r$ and we also know that $\frac{\Delta r}{r} < 0.03$, so the percentage error is:

$$\frac{\Delta A}{A} \cdot 100\% = \frac{12r \cdot \Delta r}{6r^2} \cdot 100\% = \frac{2\Delta r}{r} \cdot 100\% < 200(0.03)\% = 6\%$$

9. Computing differentials:

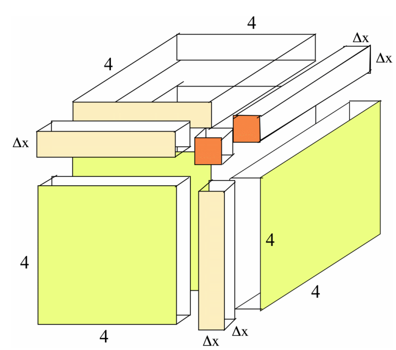
$$\begin{aligned} df &= f'(x) \cdot dx = \frac{1}{x}\,dx \\ du &= \frac{du}{dx} \cdot dx = \frac{-3}{2\sqrt{1-3x}}\,dx \\ dr &= \frac{dr}{d\theta}\,d\theta = -3\sin(\theta)\,d\theta \end{aligned}$$

10. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$ so:

$$L(4+\Delta x) = f(4) + f'(4)\Delta x = 4^3 + 3(4)^2\Delta x = 64 + 48\Delta x$$

Evaluating the various quantities at the indicated points:

Δx	$f(4+\Delta x)$	$L(4+\Delta x)$	$\lvert f(4+\Delta x) - L(4+\Delta x)\rvert$
0.1	68.921	68.8	0.121
0.05	66.430125	66.4	0.030125
0.01	64.481201	64.48	0.001201
0.001	64.048012	64.048	0.000012



$f(4+\Delta x)$ is the actual volume of the cube with side length $4+\Delta x$. $L(4+\Delta x)$ is the volume of the cube with side length 4 ($V = 64$) plus the volume of the three "slabs" ($V = 3 \cdot 4^2 \cdot \Delta x$).
$\lvert f(4+\Delta x) - L(4+\Delta x)\rvert$ is the volume of the "leftover" pieces from L: the three "rods" ($V = 3 \cdot 4 \cdot (\Delta x)^2$) and the tiny cube ($V = (\Delta x)^3$).

2.9 *Implicit and Logarithmic Differentiation*

This short section presents two more differentiation techniques, both more specialized than the ones we have already seen — and consequently used on a smaller class of functions. For some functions, however, one of these techniques may be the only method that works. The idea of each method is straightforward, but actually using each of them requires that you proceed carefully and practice.

Implicit Differentiation

In our work up until now, the functions we needed to differentiate were either given **explicitly** as a function of x, such as $y = f(x) = x^2 + \sin(x)$, or it was fairly straightforward to find an explicit formula, such as solving $y^3 - 3x^2 = 5$ to get $y = \sqrt[3]{5 + 3x^2}$. Sometimes, however, we will have an equation relating x and y that is either difficult or impossible to solve explicitly for y, such as $y^2 + 2y = \sin(x) + 4$ (difficult) or $y + \sin(y) = x^3 - x$ (impossible). In each case, we can still find $y' = f'(x)$ by using **implicit differentiation**.

The key idea behind implicit differentiation is to *assume* that y is a function of x even if we cannot explicitly solve for y. This assumption does not require any work, but we need to be very careful to treat y as a function when we differentiate and to use the Chain Rule or the Power Rule for Functions.

Example 1. Assume y is a function of x and compute each derivative:

(a) $\mathbf{D}(y^3)$ (b) $\frac{d}{dx}\left(x^3y^2\right)$ (c) $(\sin(y))'$

Solution. (a) We need the Power Rule for Functions because y is a function of x:

$$\mathbf{D}(y^3) = 3y^2 \cdot \mathbf{D}(y) = 3y^2 \cdot y'$$

(b) We need to use the Product Rule and the Chain Rule:

$$\frac{d}{dx}\left(x^3y^2\right) = x^3 \cdot \frac{d}{dx}\left(y^2\right) + y^2 \cdot \frac{d}{dx}\left(x^3\right) = x^3 \cdot 2y \cdot \frac{dy}{dx} + y^2 \cdot 3x^2$$

(c) We just need to remember that $\mathbf{D}(\sin(u)) = \cos(u)$ and then use the Chain Rule: $(\sin(y))' = \cos(y) \cdot y'$. ◀

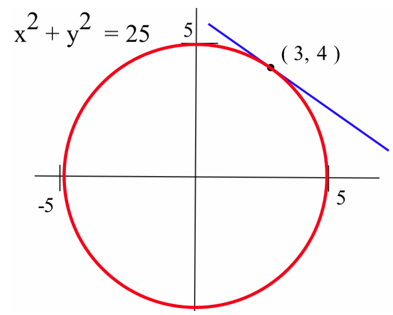
Practice 1. Assume that y is a function of x. Calculate:

(a) $\mathbf{D}\left(x^2 + y^2\right)$ (b) $\frac{d}{dx}\left(\sin(2 + 3y)\right)$.

> **Implicit Differentiation**:
>
> To determine y', differentiate each side of the defining equation, treating y as a function of x, and then algebraically solve for y'.

Example 2. Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$ with and without implicit differentiation.

Solution. Explicitly: We can solve $x^2 + y^2 = 25$ for y: $y = \pm\sqrt{25 - x^2}$ but because the point $(3, 4)$ is on the top half of the circle, we just need $y = \sqrt{25 - x^2}$ so:



$$\mathbf{D}(y) = \mathbf{D}\left(\sqrt{25 - x^2}\right) = \frac{1}{2}\left(25 - x^2\right)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{25 - x^2}}$$

Replacing x with 3, we have $y' = \frac{-3}{\sqrt{25-3^2}} = -\frac{3}{4}$.

Implicitly: We differentiate each side of the equation $x^2 + y^2 = 25$ treating y as a function of x and then solve for y':

$$\mathbf{D}\left(x^2 + y^2\right) = \mathbf{D}(25) \;\Rightarrow\; 2x + 2y \cdot y' = 0 \;\Rightarrow y' = \frac{-2x}{2y} = -\frac{x}{y}$$

so at the point $(3, 4)$, $y' = -\frac{3}{4}$, the same answer we found explicitly. ◀

Practice 2. Find the slope of the tangent line to $y^3 - 3x^2 = 15$ at the point $(2, 3)$ with and without implicit differentiation.

In the previous Example and Practice problem, it was easy to explicitly solve for y, and then we could differentiate y to get y'. Because we could explicitly solve for y, we had a choice of methods for calculating y'. Sometimes, however, we cannot explicitly solve for y and the only way to determine y' is with implicit differentiation.

Example 3. Determine y' at $(0, 2)$ for $y^2 + 2y = \sin(x) + 8$.

Solution. Assuming that y is a function of x and differentiating each side of the equation, we get:

$$\mathbf{D}\left(y^2 + 2y\right) = \mathbf{D}\left(\sin(x) + 8\right) \;\Rightarrow\; 2y \cdot y' + 2y' = \cos(x)$$

$$\Rightarrow\; (2y + 2)y' = \cos(x) \;\Rightarrow y' = \frac{\cos(x)}{2y + 2}$$

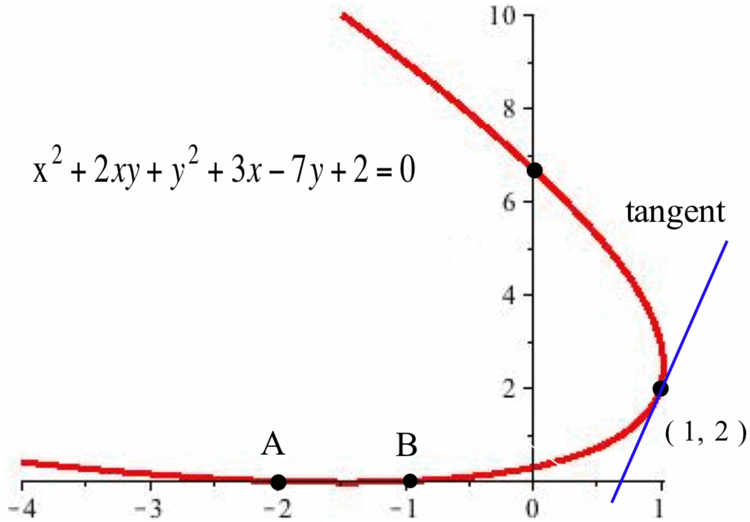
so, at the point $(0, 2)$, $y' = \frac{\cos(0)}{2(2) + 2} = \frac{1}{6}$. ◀

We could have first solved the equation explicitly for y using the quadratic formula. Do you see how? Would that make the problem easier or harder than using implicit differentiation?

Practice 3. Determine y' at $(1, 0)$ for $y + \sin(y) = x^3 - x$.

In practice, the equations may be rather complicated, but if you proceed carefully and step by step, implicit differentiation is not difficult. Just remember that y **must be treated as a function** so every time you differentiate a term containing a y you should use the Chain Rule and get something that has a y'. The algebra needed to solve for y' is always easy — if you differentiated correctly, the resulting equation will be a linear equation in the variable y'.

Example 4. Find an equation of the tangent line L to the "tilted" parabola graphed below at the point $(1,2)$.



Solution. The line goes through the point $(1,2)$ so we need to find the slope there. Differentiating each side of the equation, we get:

$$\mathbf{D}\left(x^2 + 2xy + y^2 + 3x - 7y + 2\right) = \mathbf{D}(0)$$

which yields:

$$\begin{aligned} 2x + 2x \cdot y' + 2y + 2y \cdot y' + 3 - 7y' &= 0 \\ \Rightarrow (2x + 2y - 7)y' &= -2x - 2y - 3 \\ \Rightarrow y' &= \frac{-2x - 2y - 3}{2x + 2y - 7} \end{aligned}$$

so the slope at $(1,2)$ is $m = y' = \dfrac{-2-4-3}{2+4-7} = 9$. Finally, an equation for the line is $y - 2 = 9(x-1)$ so $y = 9x - 7$. ◀

Practice 4. Find the points where the parabola graphed above crosses the y-axis, and find the slopes of the tangent lines at those points.

Implicit differentiation provides an alternate method for differentiating equations that can be solved explicitly for the function we want, and it is the *only* method for finding the derivative of a function that we cannot describe explicitly.

Logarithmic Differentiation

In Section 2.5 we saw that $D\left(\ln(f(x))\right) = \dfrac{f'(x)}{f(x)}$. If we simply multiply each side by $f(x)$, we have: $f'(x) = f(x) \cdot \mathbf{D}\left(\ln(f(x))\right)$. When the logarithm of a function is simpler than the function itself, it is often easier to differentiate the logarithm of f than to differentiate f itself.

Logarithmic Differentiation:

$$f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x)))$$

In words: The derivative of f is f times the derivative of the natural logarithm of f. Usually it is easiest to proceed in three steps:

- Calculate $\ln(f(x))$ and simplify.
- Calculate $\mathbf{D}(\ln(f(x)))$ and simplify
- Multiply the result in the previous step by $f(x)$.

Let's examine what happens when we use this process on an "easy" function, $f(x) = x^2$, and a "hard" one, $f(x) = 2^x$. Certainly we don't need to use logarithmic differentiation to find the derivative of $f(x) = x^2$, but sometimes it is instructive to try a new algorithm on a familiar function. Logarithmic differentiation is the easiest way to find the derivative of $f(x) = 2^x$ (if we don't remember the pattern for differentiating a^x from Section 2.5).

$f(x) = x^2$
$\ln(f(x)) = \ln(x^2) = 2 \cdot \ln(x)$
$\mathbf{D}(\ln(f(x))) = \mathbf{D}(2 \cdot \ln(x)) = \frac{2}{x}$
$f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x))) = x^2 \cdot \frac{2}{x} = 2x$

$f(x) = 2^x$
$\ln(f(x)) = \ln(2^x) = x \cdot \ln(2)$
$\mathbf{D}(\ln(f(x))) = \mathbf{D}(x \cdot \ln(2)) = \ln(2)$
$f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x))) = 2^x \cdot \ln(2)$

Example 5. Use the pattern $f'(x) = f(x) \cdot \mathbf{D}(\ln(f(x)))$ to find the derivative of $f(x) = (3x+7)^5 \sin(2x)$.

Solution. Apply the natural logarithm to both sides and rewrite:

$$\ln(f(x)) = \ln\left((3x+7)^5 \cdot \sin(2x)\right) = 5\ln(3x+7) + \ln(\sin(2x))$$

so:

$$\begin{aligned} \mathbf{D}(\ln(f(x))) &= \mathbf{D}(5\ln(3x+7) + \ln(\sin(2x))) \\ &= 5 \cdot \frac{3}{3x+7} + 2 \cdot \frac{\cos(2x)}{\sin(2x)} \end{aligned}$$

Then:

$$\begin{aligned} f'(x) &= f(x) \cdot \mathbf{D}(\ln(f(x))) \\ &= (3x+7)^5 \sin(2x) \left(\frac{15}{3x+7} + 2 \cdot \frac{\cos(2x)}{\sin(2x)} \right) \\ &= 15(3x+7)^4 \sin(2x) + 2(3x+7)^5 \cos(2x) \end{aligned}$$

the same result we would obtain using the Product Rule. ◀

Practice 5. Use logarithmic differentiation to find the derivative of $f(x) = (2x+1)^3(3x^2-4)^7(x+7)^4$.

We could have differentiated the functions in the previous Example and Practice problem without logarithmic differentiation. There are, however, functions for which logarithmic differentiation is the **only** method we can use. We know how to differentiate x raised to a constant power, $\mathbf{D}(x^p) = p \cdot x^{p-1}$, and a constant to a variable power, $\mathbf{D}(b^x) = b^x \ln(b)$, but the function $f(x) = x^x$ has both a variable base and a variable power, so neither differentiation rule applies. We need to use logarithmic differentiation.

Example 6. Find $\mathbf{D}(x^x)$, assuming that $x > 0$.

Solution. Apply the natural logarithm to both sides and rewrite:

$$\ln(f(x)) = \ln(x^x) = x \cdot \ln(x)$$

so:

$$\begin{aligned}\mathbf{D}(\ln(f(x))) &= \mathbf{D}(x \cdot \ln(x)) = x \cdot \mathbf{D}(\ln(x)) + \ln(x) \cdot \mathbf{D}(x) \\ &= x \cdot \frac{1}{x} + \ln(x) \cdot 1 = 1 + \ln(x)\end{aligned}$$

Then $\mathbf{D}(x^x) = f'(x) = f(x)\,\mathbf{D}(\ln(f(x))) = x^x(1 + \ln(x))$. ◀

Practice 6. Find $\mathbf{D}\left(x^{\sin(x)}\right)$ assuming that $x > 0$.

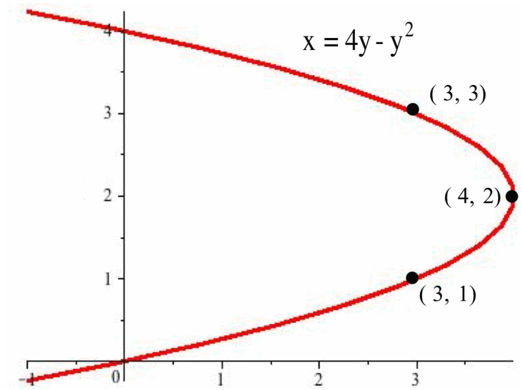
Logarithmic differentiation is an alternate method for differentiating some functions such as products and quotients, and it is the only method we've seen for differentiating some other functions such as variable bases to variable exponents.

2.9 Problems

In Problems 1–10 find $\frac{dy}{dx}$ in two ways: (a) by differentiating implicitly and (b) by explicitly solving for y and then differentiating. Then find the value of $\frac{dy}{dx}$ at the given point using your results from both the implicit and the explicit differentiation.

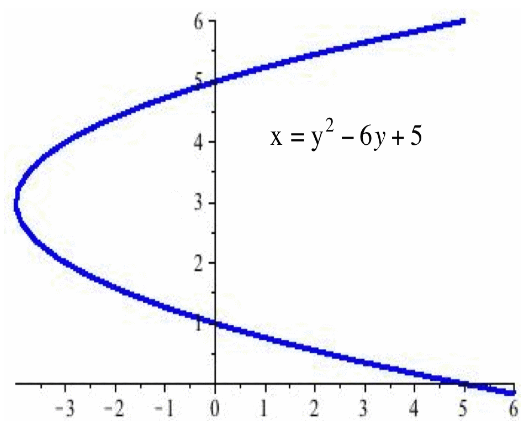
1. $x^2 + y^2 = 100$, point: $(6, 8)$
2. $x^2 + 5y^2 = 45$, point: $(5, 2)$
3. $x^2 - 3xy + 7y = 5$, point: $(2, 1)$
4. $\sqrt{x} + \sqrt{y} = 5$, point: $(4, 9)$
5. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, point: $(0, 4)$
6. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, point: $(3, 0)$
7. $\ln(y) + 3x - 7 = 0$, point: $(2, e)$
8. $x^2 - y^2 = 16$, point: $(5, 3)$
9. $x^2 - y^2 = 16$, point: $(5, -3)$
10. $y^2 + 7x^3 - 3x = 8$, point: $(1, 2)$

11. Find the slopes of the lines tangent to the graph below at the points $(3,1)$, $(3,3)$ and $(4,2)$.



12. Find the slopes of the lines tangent to the graph in the figure above at the points where the graph crosses the y-axis.

13. Find the slopes of the lines tangent to the graph below at the points $(5,0)$, $(5,6)$ and $(-4,3)$.



14. Find the slopes of the lines tangent to the graph in the figure above at the points where the graph crosses the y-axis.

In Problems 15–22 , find $\frac{dy}{dx}$ using implicit differentiation and then find the slope of the line tangent to the graph of the equation at the given point.

15. $y^3 - 5y = 5x^2 + 7$, point: $(1,3)$

16. $y^2 - 5xy + x^2 + 21 = 0$, point: $(2,5)$

17. $y^2 + \sin(y) = 2x - 6$, point: $(3,0)$

18. $y + 2x^2y^3 = 4x + 7$, point: $(3,1)$

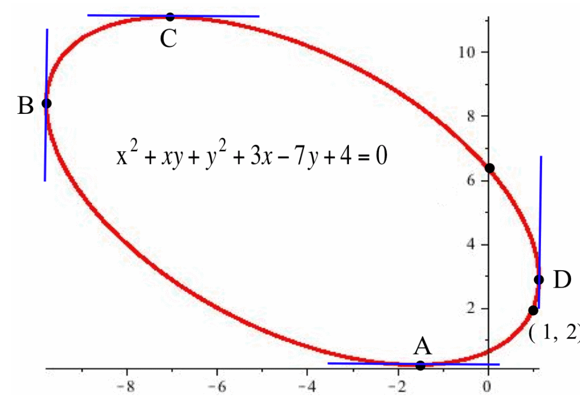
19. $e^y + \sin(y) = x^2 - 3$, point: $(2,0)$

20. $\left(x^2 + y^2 + 1\right)^2 - 4x^2 = 81$, point: $(0, 2\sqrt{2})$

21. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$, point: $(8,1)$

22. $x + \cos(xy) = y + 3$, point: $(2,0)$

23. Find the slope of the line tangent to the ellipse shown in the figure below at the point $(1,2)$.



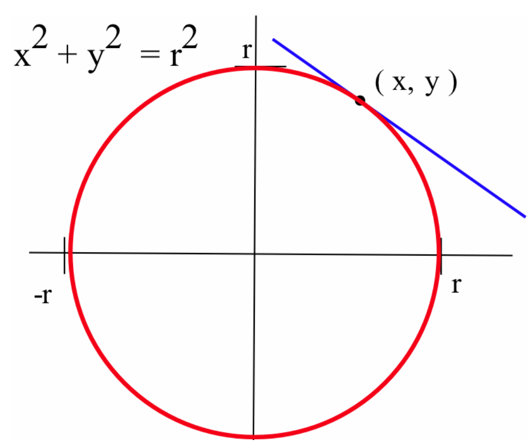
24. Find the slopes of the tangent lines at the points where the ellipse shown above crosses the y-axis.

25. Find y' for $y = Ax^2 + Bx + C$ and for the equation $x = Ay^2 + By + C$.

26. Find y' for $y = Ax^3 + B$ and for $x = Ay^3 + B$.

27. Find y' for $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

28. In Chapter 1 we assumed that the tangent line to a circle at a point was perpendicular to the radial line passing through that point and the center of the circle. Use implicit differentiation to prove that the line tangent to the circle $x^2 + y^2 = r^2$ (see below) at an arbitrary point (x, y) is perpendicular to the line passing through $(0,0)$ and (x, y).



Problems 29–31 use the figure from Problems 23–24.

29. Find the coordinates of point A where the tangent line to the ellipse is horizontal.

30. Find the coordinates of point B where the tangent line to the ellipse is vertical.

31. Find the coordinates of points C and D.

In 32–40, find $\frac{dy}{dx}$ in two ways: (a) by using the "usual" differentiation patterns and (b) by using logarithmic differentiation.

32. $y = x \cdot \sin(3x)$

33. $y = (x^2+5)^7(x^3-1)^4$

34. $y = \dfrac{\sin(3x-1)}{x+7}$

35. $y = x^5 \cdot (3x+2)^4$

36. $y = 7^x$

37. $y = e^{\sin(x)}$

38. $y = \cos^7(2x+5)$

39. $y = \sqrt{25-x^2}$

40. $y = \dfrac{x \cdot \cos(x)}{x^2+1}$

In 41–46, use logarithmic differentiation to find $\frac{dy}{dx}$.

41. $y = x^{\cos(x)}$

42. $y = (\cos(x))^x$

43. $y = x^4 \cdot (x-2)^7 \cdot \sin(3x)$

44. $y = \dfrac{\sqrt{x+10}}{(2x+3)^3 \cdot (5x-1)^7}$

45. $y = (3+\sin(x))^x$

46. $y = \sqrt{\dfrac{x^2+1}{x^2-1}}$

In 47–50, use the values in each table to calculate the values of the derivative in the last column.

47.

x	$f(x)$	$\ln(f(x))$	$\mathbf{D}(\ln(f(x)))$	$f'(x)$
1	1	0.0	1.2	
2	9	2.2	1.8	
3	64	4.2	2.1	

48.

x	$g(x)$	$\ln(g(x))$	$\mathbf{D}(\ln(g(x)))$	$g'(x)$
1	5	1.6	0.6	
2	10	2.3	0.7	
3	20	3.0	0.8	

49.

x	$f(x)$	$\ln(f(x))$	$\mathbf{D}(\ln(f(x)))$	$f'(x)$
1	5	1.6	-1	
2	2	0.7	0	
3	7	1.9	2	

50.

x	$g(x)$	$\ln(g(x))$	$\mathbf{D}(\ln(g(x)))$	$g'(x)$
2	1.4	0.3	1.2	
3	3.3	1.2	0.6	
7	13.6	2.6	0.2	

Problems 51–55 illustrate how logarithmic differentiation can be used to verify some differentiation patterns we already know (51–52, 54) and to derive some new patterns (53, 55). Assume that all of the functions are differentiable and that the function combinations are defined.

51. Use logarithmic differentiation on $f \cdot g$ to re-derive the Product Rule: $\mathbf{D}(f \cdot g) = f \cdot g' + g \cdot f'$.

52. Use logarithmic differentiation on $\dfrac{f}{g}$ to re-derive the quotient rule: $\mathbf{D}\left(\dfrac{f}{g}\right) = \dfrac{g \cdot f' - f \cdot g'}{g^2}$.

53. Use logarithmic differentiation to obtain a product rule for three functions: $\mathbf{D}(f \cdot g \cdot h) = ?$.

54. Use logarithmic differentiation on the exponential function a^x (with $a > 0$) to show that its derivative is $a^x \ln(a)$.

55. Use logarithmic differentiation to determine a pattern for the derivative of f^g: $\mathbf{D}(f^g) = ?$.

56. In Section 2.1 we proved the Power Rule $\mathbf{D}(x^n) = n \cdot x^{n-1}$ for any positive integer n.
    (a) Why does this formula hold for $n = 0$?
    (b) Use the Quotient Rule to prove that $\mathbf{D}(x^{-m}) = -m \cdot x^{-m-1}$ for any positive integer m and conclude that the Power Rule holds for all integers.
    (c) Now let $y = x^{\frac{p}{q}}$ where p and q are integers so that $y^q = x^p$. Use implicit differentiation to show that the Power Rule holds for all rational exponents. (We still have not considered the case where $y = x^a$ with a an irrational number, because we haven't actually *defined* what x^a means for a irrational. We will take care of that—and the extension of the Power Rule to all real exponents—in Chapter 7.)

2.9 Practice Answers

1. $\mathbf{D}(x^2+y^2) = 2x + 2y \cdot y'$
   $\frac{d}{dx}(\sin(2+3y)) = \cos(2+3y) \cdot \mathbf{D}(2+3y) = \cos(2+3y) \cdot 3y'$

2. Explicitly: $y' = \dfrac{1}{3}\left(3x^2+15\right)^{-\frac{2}{3}} \mathbf{D}\left(3x^2+15\right) = \dfrac{1}{3}\left(3x^2+15\right)^{-\frac{2}{3}}(6x)$.
   When $(x,y) = (2,3)$, $y' = \frac{1}{3}\left(3(2)^2+15\right)^{\frac{2}{3}}(6\cdot 2) = 4\,(27)^{-\frac{2}{3}} = \frac{4}{9}$.
   Implicitly: $\mathbf{D}\left(y^3 - 3x^2\right) = \mathbf{D}(15) \Rightarrow 3y^2 \cdot y' - 6x = 0$ so $y' = \frac{2x}{y^2}$.
   When $(x,y) = (2,3)$, $y' = \dfrac{2\cdot 2}{3^2} = \dfrac{4}{9}$.

3. $\mathbf{D}(y+\sin(y)) = \mathbf{D}\left(x^3-x\right) \Rightarrow y' + \cos(y)\cdot y' = 3x^2 - 1 \Rightarrow y' \cdot (1+\cos(y)) = 3x^2-1$, so we have $y' = \frac{3x^2-1}{1+\cos(y)}$. When $(x,y) = (1,0)$, $y' = \dfrac{3(1)^2-1}{1+\cos(0)} = 1$.

4. To find where the parabola crosses the y-axis, we can set $x=0$ and solve for the values of y: $y^2 - 7y + 2 = 0 \Rightarrow y = \frac{7\pm\sqrt{(-7)^2-4(1)(2)}}{2(1)} = \frac{7\pm\sqrt{41}}{2} \approx 0.3$ and 6.7. The parabola crosses the y-axis (approximately) at the points $(0, 0.3)$ and $(0, 6.7)$. From Example 4, we know that $y' = \dfrac{-2x-2y-3}{2x+2y-7}$, so at the point $(0, 0.3)$, the slope is approximately $\dfrac{0-0.6-3}{0+0.6-7} \approx 0.56$, and at the point $(0, 6.7)$, the slope is approximately $\dfrac{0-13.4-3}{0+13.4-7} \approx -2.56$.

5. Applying the formula $f'(x) = f(x)\cdot\mathbf{D}(\ln(f(x)))$ to the function $f(x) = (2x+1)^3(3x^2-4)^7(x+7)^4$, we have:
   $$\ln(f(x)) = 3\cdot\ln(2x+1) + 7\cdot\ln(3x^2-4) + 4\cdot\ln(x+7)$$
   so:
   $$\mathbf{D}(\ln(f(x))) = \frac{3}{2x+1}(2) + \frac{7}{3x^2-4}(6x) + \frac{4}{x+7}(1)$$
   and thus:
   $$f'(x) = f(x)\cdot\mathbf{D}(\ln(f(x))) = (2x+1)^3(3x^2-4)^7(x+7)^4 \cdot \left[\frac{6}{2x+1} + \frac{42x}{3x^2-4} + \frac{4}{x+7}\right]$$

6. Using $f'(x) = f(x)\cdot\mathbf{D}(\ln(f(x)))$ with $f(x) = x^{\sin(x)}$:
   $$\ln(f(x)) = \ln\left(x^{\sin(x)}\right) = \sin(x)\cdot\ln(x)$$
   so:
   $$D(\ln(f(x))) = \mathbf{D}(\sin(x)\cdot\ln(x)) = \sin(x)\cdot\mathbf{D}(\ln(x)) + \ln(x)\cdot\mathbf{D}(\sin(x)) = \sin(x)\cdot\frac{1}{x} + \ln(x)\cdot\cos(x)$$
   and thus:
   $$f'(x) = f(x)\cdot\mathbf{D}(\ln(f(x))) = x^{\sin(x)}\cdot\left[\frac{\sin(x)}{x} + \ln(x)\cdot\cos(x)\right]$$