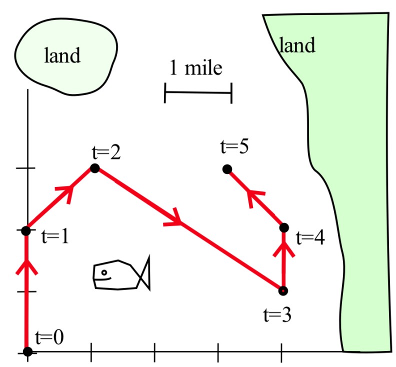
5.3 *Arclength and Surface Area*

This section introduces two additional geometric applications of integration: finding the length of a curve and finding the area of a surface generated when you revolve a curve about a line. The general strategy remains the same: partition the problem into small pieces, approximate the solution on each small piece, add the small solutions together to form a Riemann sum and, finally, take the limit of the Riemann sum to get a definite integral.

Arclength: How Long Is a Curve?

In order to better understand an animal, biologists need to know how it moves through its environment and how far it travels. We need to know the length of the path it moves along. If we know the object's location at successive times, then we can easily calculate the distances between those locations and add them together to get a total (approximate) distance.



Example 1. In order to study the movement of whales, marine biologists implant a small transmitter on selected whales and track the location of a whale via satellite. Position data at one-hour time intervals over a five-hour period appears in the margin figure. How far did the whale swim during the first three hours?

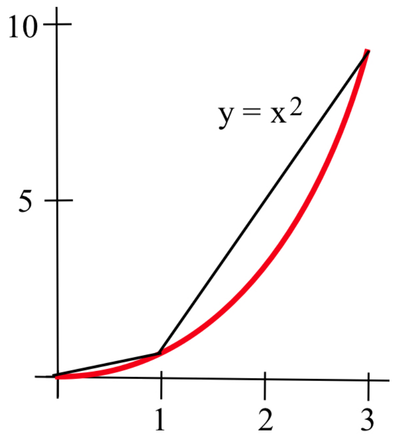
Solution. In moving from the point $(0,0)$ to the point $(0,2)$, the whale traveled *at least* 2 miles. Similarly, the whale traveled at least $\sqrt{(1-0)^2+(3-2)^2} = \sqrt{2} \approx 1.4$ miles during the second hour and at least $\sqrt{(4-1)^2+(1-3)^2} = \sqrt{13} \approx 3.6$ miles during the third hour. The scientist concluded that the whale swam at least $2+1.4+3.6 = 7$ miles during the three-hour period. ◀

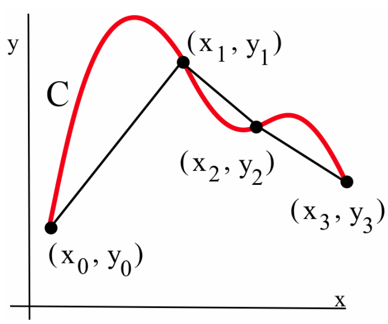
Practice 1. How far did the whale swim during the entire five-hour time period?

It is unlikely that the whale swam in a straight line from location to location, so its actual swimming distance was undoubtedly more than seven miles during the first three hours. Scientists might get better distance estimates by recording the whale's position over shorter, five-minute time intervals.

Our strategy for finding the length of a curve will resemble the one the scientist used, and if the locations are given by a formula, then we can calculate the successive locations over very short intervals and get very good approximations of the total path length.

Example 2. Use the points $(0,0)$, $(1,1)$ and $(3,9)$ to approximate the length of $y = x^2$ for $0 \le x \le 3$.





Solution. The lengths of the two line segments (see margin) are:

$$\sqrt{(1-0)^2+(1-0)^2} = \sqrt{1+1} = \sqrt{2} \approx 1.41$$

and:

$$\sqrt{(3-1)^2+(9-1)^2} = \sqrt{4+64} = \sqrt{68} \approx 8.25$$

so the length of the curve is approximately $1.41 + 8.25 = 9.66$. ◀

Practice 2. Get a better approximation of the length of $y = x^2$ for $0 \le x \le 3$ by using the points $(0,0)$, $(1,1)$, $(2,4)$ and $(3,9)$. Is your approximation longer or shorter than the actual length?

For a curve $\mathcal{C}$ (see margin), pick some points (x_k, y_k) along $\mathcal{C}$ and connect those points with line segments. Then the sum of the lengths of the line segments will approximate the length of $\mathcal{C}$. We can think of this as pinning a string to the curve at the selected points, and then measuring the length of the string as an approximation of the length of the curve. Of course, if we only pick a few points (as in the margin), then the total length approximation will probably be rather poor, so eventually we want lots of points (x_k, y_k) close together all along $\mathcal{C}$.

Label these points so that (x_0, y_0) is one endpoint of $\mathcal{C}$ and (x_n, y_n) is the other endpoint, and so that the subscripts increase as we move along $\mathcal{C}$. Then the distance between the successive points (x_{k-1}, y_{k-1}) and (x_k, y_k) is:

$$\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

and the total length of these line segments is simply the sum of the successive lengths:

$$\sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

This summation does not have the form $\sum g(c_k) \cdot \Delta x_k$ so it is not a Riemann sum. It is, however, algebraically equivalent to an expression very much like a Riemann sum that will lead us to a definite integral representation for the length of $\mathcal{C}$.

If $\mathcal{C}$ is given by $y = f(x)$ for $a \le x \le b$, so that y is a function of x, we can factor $(\Delta x_k)^2$ from inside the radical and simplify:

$$\text{length of } C \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 \left[1 + \frac{(\Delta y_k)^2}{(\Delta x_k)^2}\right]}$$

$$= \sum_{k=1}^{n} (\Delta x_k) \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} = \sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k$$

to get an expression that looks more like a Riemann sum. The $\dfrac{\Delta y_k}{\Delta x_k}$ inside the radical should remind you of two things: the slope of a line segment (it is, in fact, the slope of the k-th line segment in our approximation of the curve $\mathcal{C}$) and a derivative, $\frac{dy}{dx}$. If $f(x)$ is both continuous and differentiable, then the Mean Value Theorem guarantees that there is some number c_k between x_{k-1} and x_k so that:

Review Section 3.2 if you need to refresh your memory about the hypotheses and conclusions of the Mean Value Theorem.

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta y_k}{\Delta x_k}$$

in which case we can write:

$$\sum_{k=1}^{n} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \cdot \Delta x_k = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta x_k$$

This last expression *is* a Riemann sum, so it converges to a definite integral:

$$\sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta x_k \longrightarrow \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

In order to be sure that the sum converges to the integral, we need the resulting integrand to be an integrable function. If we require $f'(x)$ to be continuous on $[a, b]$ then the integrand will be a composition of continuous functions, hence continuous, and we know that a function that is continuous on a closed interval is integrable.

This definite integral provides us with a formula for the length of a curve $\mathcal{C}$ given by $y = f(x)$ for $a \le x \le b$.

Arclength Formula: $y = f(x)$ version

If	$\mathcal{C}$ is a curve given by $y = f(x)$ for $a \le x \le b$
and	$f'(x)$ exists and is continuous on $[a, b]$
then	the length L of $\mathcal{C}$ is given by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example 3. Compute the length of $y = x^2$ for $0 \le x \le 3$.

Solution. Here $f(x) = x^2 \Rightarrow f'(x) = 2x$ so the length of this curve is:

$$\int_0^3 \sqrt{1 + [2x]^2}\, dx = \int_0^3 \sqrt{1 + 4x^2}\, dx$$

Unfortunately we do not (yet) have a technique to find an antiderivative of this integrand, but we can use numerical methods (such as Simpson's Rule, or a calculator or computer) to determine that the value of the integral is approximately 9.7471 (compare this with the answers from Example 2 and Practice 2). ◀

You will eventually be able to find an exact value for this definite integral using techniques developed in Section 8.4.

Practice 3. Compute the length of $y = x^2$ between $(1, 1)$ and $(4, 16)$.

Practice 4. Represent the length of one period of $y = \sin(x)$ as a definite integral, then find the length of this curve (using technology to approximate the value of the definite integral, if necessary).

More Arclength Formulas

Not all interesting curves are graphs of functions of the form $y = f(x)$. For a curve given by $x = g(y)$ we can mimic the previous argument (or simply swap x and y) to arrive at another arclength formula:

Arclength Formula: $x = g(y)$ version

If	$\mathcal{C}$ is a curve given by $x = g(y)$ for $c \le y \le d$
and	$g'(y)$ exists and is continuous on $[c, d]$
then	the length L of $\mathcal{C}$ is given by:

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Practice 5. Compute the length of $x = \sqrt{y}$ between $(1, 1)$ and $(4, 16)$.

Review Section 2.5 to refresh your memory about parametric equations.

A curve $\mathcal{C}$ can also be described using parametric equations, where functions $x(t)$ and $y(t)$ give the coordinates of a point on the curve specified by a parameter t. We often think of t as "time," so that $(x(t), y(t))$ represents the position of a particle in the xy-plane t seconds (or minutes or hours) after time $t = 0$. In Section 2.5, we discovered that the speed of such a particle at time t is given by:

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

The distance traveled by the particle and the length of the curve will be equal as long as the particle does not traverse any part of the curve more than once on the interval $\alpha \le t < \beta$.

To find the distance the particle travels between times $t = \alpha$ and $t = \beta$, we could then integrate this speed function, which would also tell us the length of the curve.

Arclength Formula (Parametric Version)

If	$\mathcal{C}$ is a curve given by $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$
and	$x'(t)$ and $y'(t)$ exist and are continuous on $[\alpha, \beta]$
then	the length L of $\mathcal{C}$ is given by:

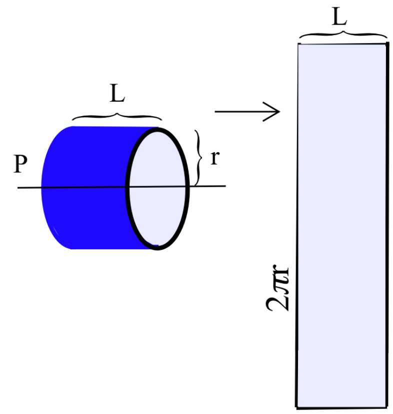
$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Practice 6. Compute the length of the parametric curve given by the functions $x(t) = \cos(t)$ and $y(t) = \sin(t)$ for $0 \le t \le 2\pi$.

Practice 7. Compute the length of the parametric path given by the functions $x(t) = 1 + 3t$ and $y(t) = 4t$ for $1 \le t \le 3$.

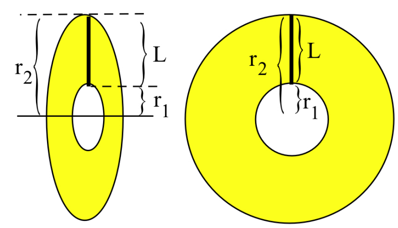
Areas of Surfaces of Revolution

In the previous section, we revolved a *region* in the xy-plane about a horizontal or vertical axis to create a *solid*, then used an integral to compute the volume of that solid. If we instead rotate a *curve* about an axis, we get a *surface*, whose **surface area** we can also compute using an integral. Just as the integral formulas for arclength came from the simple distance formula, the integral formulas for the area of a surface of revolution come from the formula for revolving a single line segment.



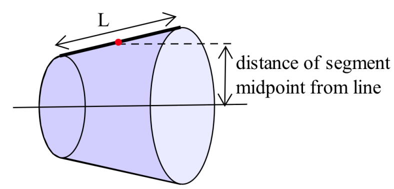
If we rotate a line segment of length L parallel to a line P (see margin) about the line P, then the resulting surface (a cylinder) can be "unrolled" and laid flat. This flattened surface is a rectangle with area $A = 2\pi \cdot r \cdot L$.

If we rotate a line segment of length L perpendicular to a line P and not intersecting P (see second margin figure) about the line P, then the resulting surface is the region between two concentric circles (an "annulus") and its area is:



$$\begin{aligned} A &= (\text{area of large circle}) - (\text{area of small circle}) \\ &= \pi (r_2)^2 - \pi (r_1)^2 = \pi \left[(r_2)^2 - (r_1)^2 \right] = \pi (r_2 + r_1)(r_2 - r_1) \\ &= 2\pi \left(\frac{r_2 + r_1}{2} \right) L \end{aligned}$$

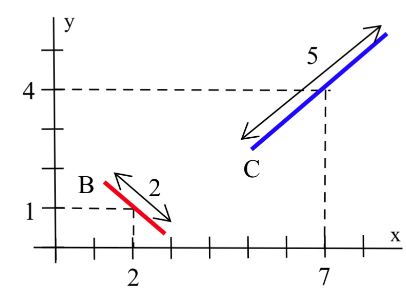
The expression $\frac{r_2+r_1}{2}$ represents the distance of the *midpoint* of the line segment L from the axis of rotation P and $2\pi\left(\frac{r_2+r_1}{2}\right)$ is the *distance* this midpoint travels when we revolve the line segment about the axis. It turns out that this pattern holds when we revolve *any* line segment of length L that does not intersect a line P about the line P (see margin):



See Problem 52 for a proof.

$$\begin{aligned} A &= (\text{distance traveled by segment midpoint}) \cdot (\text{length of line segment}) \\ &= 2\pi\, (\text{distance of segment midpoint from line } P) \cdot L \end{aligned}$$

Example 4. Compute the area of the surface generated when each line segment in the margin figure is rotated about the x-axis and the y-axis.



Solution. Line segment B has length $L = 2$ and its midpoint is at $(2, 1)$, which is 1 unit from the x-axis and 2 units from the y-axis. When B is rotated about the x-axis, the surface area is therefore:

$$2\pi \cdot (\text{distance of midpoint from } x\text{-axis}) \cdot 2 = 2\pi(1)2 = 4\pi$$

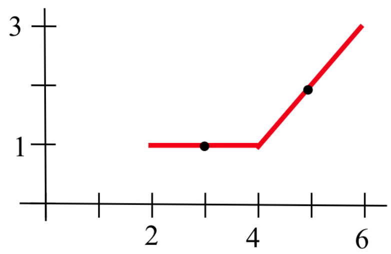
and when B is rotated about the y-axis, the surface area is:

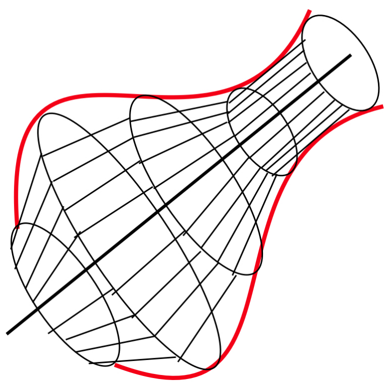
$$2\pi \cdot (\text{distance of midpoint from } y\text{-axis}) \cdot 2 = 2\pi(2)2 = 8\pi$$

Line segment $\mathcal{C}$ has length 5 and its midpoint is at $(7, 4)$. When $\mathcal{C}$ is rotated about the x-axis, the resulting surface area is:

$$2\pi \cdot (\text{distance of midpoint from } x\text{-axis}) \cdot 5 = 2\pi(4)5 = 40\pi$$

When $\mathcal{C}$ is rotated about the y-axis, the distance of the midpoint from the axis is 7, so the surface area is $2\pi(7)5 = 70\pi$. ◀





Practice 8. Find the area of the surface generated when the graph in the margin is rotated about each coordinate axis.

When we rotate a *curve* $\mathcal{C}$ (that does not intersect a line P, as in the second margin figure) about the line P, we also get a surface. To approximate the area of that surface, we can use the same strategy we used to approximate the length of a curve: select some points (x_k, y_k) along the curve, connect the points with line segments, calculate the surface area of each rotated line segment, and add together the surface areas of the rotated line segments.

The rotated line segment with endpoints (x_{k-1}, y_{k-1}) and (x_k, y_k) has midpoint:

$$(\overline{x}_k, \overline{y}_k) = \left(\frac{x_{k-1}+x_k}{2}, \frac{y_{k-1}+y_k}{2}\right)$$

and length:

$$L = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

If we rotate $\mathcal{C}$ about the x-axis, the distance from the midpoint of the k-th line segment to the x-axis is $\overline{y}_k$ so the surface area of the k-th rotated line segment will be:

$$2\pi\left(\overline{y}_k\right)L = 2\pi\left(\frac{y_{k-1}+y_k}{2}\right)\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$
$$= 2\pi\left(\frac{y_{k-1}+y_k}{2}\right)\sqrt{1+\left[\frac{\Delta y_k}{\Delta x_k}\right]^2}\,\Delta x_k$$

If $\mathcal{C}$ is given by $y = f(x)$ for $a \le x \le b$, and $f'(x)$ is continuous on $[a, b]$, we can appeal to the Mean Value Theorem to find a c_k with $x_{k-1} < c_k < x_k$ and $f'(c_k) = \frac{\Delta y_k}{\Delta x_k}$ so that our last expression becomes:

$$2\pi\left(\frac{f(x_{k-1})+f(x_k)}{2}\right)\sqrt{1+\left[f'(c_k)\right]^2}\,\Delta x_k$$

Adding up these approximations, we get:

This is not actually a Riemann sum, because the values x_{k-1}, x_k and c_k are not all the same. Proving that this sum converges to a definite integral requires some more advanced techniques.

$$\sum_{k=1}^{n} 2\pi\left(\frac{f(x_{k-1})+f(x_k)}{2}\right)\sqrt{1+\left[f'(c_k)\right]^2}\,\Delta x_k$$

which converges to a definite integral:

$$\int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

that gives us a formula for the surface area of the revolved curve.

Example 5. Compute the area of the surface generated when the curve $y = 2 + x^2$ for $0 \le x \le 3$ is rotated about the x-axis.

Solution. Here $f(x) = 2 + x^2 \Rightarrow f'(x) = 2x$ so, using the integral formula we just obtained:

$$\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx = \int_0^3 2\pi\left(2+x^2\right)\sqrt{1+4x^2}\,dx$$

We do not (yet) know how to find an antiderivative for this integrand, but numerical approximation yields a result of 383.8. ◀

You will eventually be able to find an exact value for this definite integral using techniques developed in Section 8.4.

More Sufrace Area Formulas

If a curve $\mathcal{C}$ given by $y = f(x)$ for $a \le x \le b$ is instead rotated about the y-axis, then the distance from the midpoint of the k-th line segment to the axis of rotation is $\overline{x}_k$. Replacing $\overline{y}_k$ with $\overline{x}_k$ in our work on the previous page yields the formula:

$$\int_a^b 2\pi x\sqrt{1+[f'(x)]^2}\,dx$$

for the area of the surface generated by revolving $\mathcal{C}$ about the y-axis (assuming, as before, that $f'(x)$ is continuous for $a \le x \le b$).

Example 6. Compute the area of the surface generated when the curve $y = 2 + x^2$ for $0 \le x \le 3$ is rotated about the y-axis.

Solution. Here again $f(x) = 2 + x^2 \Rightarrow f'(x) = 2x$ so, using our newest integral formula:

$$\int_a^b 2\pi x\sqrt{1+[f'(x)]^2}\,dx = \int_0^3 2\pi x\sqrt{1+4x^2}\,dx$$

We can find an antiderivative of this integrand using substitution:

$$u = 1 + 4x^2 \Rightarrow du = 8x\,dx \Rightarrow \frac{1}{8}\,du = x\,dx$$

The integral limits become $u = 1 + 4(0)^2 = 1$ and $u = 1 + 4(3)^2 = 37$, so the surface area is:

$$\int_{u=1}^{u=37} 2\pi\cdot\frac{1}{8}\sqrt{u}\,du = \frac{\pi}{4}\int_1^{37} u^{\frac{1}{2}}\,du = \frac{\pi}{4}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{37} = \frac{\pi}{6}\left[37\sqrt{37}-1\right]$$

or approximately 117.3. ◀

Wrap-Up

Developing formulas for the area of a surface generated by rotating a curve $x = g(y)$ for $c \le y \le d$ (or by parametric equations) present little additional difficulty. In future chapters, however, we will develop much more general—yet simpler—formulas for arclength and surface area. While the integral formulas developed in this section can be useful, more importantly their development served to illustrate yet again how relatively simple approximation formulas can lead us—via Riemann sums—to integral formulas. We will see this process again and again.

See Problems 48–51.

5.3 Problems

1. The locations (in feet, relative to an oak tree) at various times (in minutes) for a squirrel spotted in a back yard appear in the table below:

time	north	east
0	10	7
5	25	27
10	1	45
15	13	33
20	24	40
25	10	23
30	0	14

At least how far did the squirrel travel during the first 15 minutes?

2. The squirrel in the previous problem traveled at least how far during the first 30 minutes?
3. Use the partition $\{0,1,2\}$ to estimate the length of $y = 2^x$ between the points $(0,1)$ and $(2,4)$.
4. Use the partition $\{1,2,3,4\}$ to estimate the length of $y = \dfrac{1}{x}$ between the points $(1,1)$ and $\left(4, \dfrac{1}{4}\right)$.

The graphs of the functions in Problems 5–8 are line segments. Calculate each length (a) using the distance formula between two points and (b) by setting up and evaluating an appropriate arclength integral.

5. $y = 1 + 2x$ for $0 \le x \le 2$.
6. $y = 5 - x$ for $1 \le x \le 4$,
7. $x = 2 + t$, $y = 1 - 2t$ for $0 \le t \le 3$.
8. $x = -1 - 4t$, $y = 2 + t$ for $1 \le t \le 4$.
9. Calculate the length of $y = \dfrac{2}{3}x^{\frac{3}{2}}$ for $0 \le x \le 4$.
10. Calculate the length of $y = 4x^{\frac{3}{2}}$ for $1 \le x \le 9$.

Very few functions of the form $y = f(x)$ lead to integrands of the form $\sqrt{1 + [f'(x)]^2}$ that have elementary antiderivatives. In 11—14, $1 + \left[f'(x)\right]^2$ ends up being a perfect square, so you can evaluate the resulting arclength integral using antiderivatives.

11. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ for $1 \le x \le 5$.
12. $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ for $1 \le x \le 9$.
13. $y = \dfrac{x^5}{5} + \dfrac{1}{12x^3}$ for $1 \le x \le 5$.
14. $y = \dfrac{x^6}{6} + \dfrac{1}{16x^4}$ for $4 \le x \le 25$.

In Problems 15–23, represent each length as a definite integral, then evaluate the integral (using technology, if necessary).

15. The length of $y = x^2$ from $(0,0)$ to $(1,1)$.
16. The length of $y = x^3$ from $(0,0)$ to $(1,1)$.
17. The length of $y = \sqrt{x}$ from $(1,1)$ to $(9,3)$.
18. The length of $y = \ln(x)$ from $(1,0)$ to $(e,1)$.
19. The length of $y = \sin(x)$ from $(0,0)$ to $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$ and from $\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$ to $\left(\dfrac{\pi}{2}, 1\right)$.
20. The length of the ellipse $x(t) = 3\cos(t)$, $y(t) = 4\sin(t)$ for $0 \le t \le 2\pi$.
21. The length of the ellipse $x(t) = 5\cos(t)$, $y(t) = 2\sin(t)$ for $0 \le t \le 2\pi$.
22. A robot programmed to be at location $x(t) = t\cos(t)$, $y(t) = t\sin(t)$ at time t will travel how far between $t = 0$ and $t = 2\pi$?
23. How far will the robot in the previous problem travel between $t = 10$ and $t = 20$?
24. As a tire of radius R rolls, a pebble stuck in the tread will travel a "cycloid" path, given by $x(t) = R \cdot (t - \sin(t))$, $y(t) = R \cdot (1 - \cos(t))$. As t increases from 0 to 2π, the tire makes one complete revolution and travels forward $2\pi R$ units. How far does the pebble travel?
25. Referring to the previous problem, as a tire with a 1-foot radius rolls forward 1 mile, how far does a pebble stuck in the tire tread travel?
26. Graph $y = x^n$ for $n = 1$, 3, 10 and 20. As the value of n becomes large, what happens to the graph of $y = x^n$? Estimate the value of:

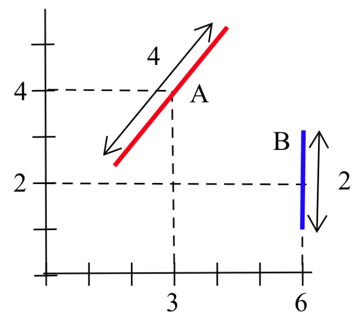
$$\lim_{n \to \infty} \int_{x=0}^{x=1} \sqrt{1 + [n \cdot x^{n-1}]^2}\, dx$$

27. Find the point on the curve $f(x) = x^2$ for $0 \le x \le 4$ that will divide the curve into two equally long pieces. Find the points that will divide the segment into three equally long pieces.

28. Find the pattern for the functions in Problems 11–14. If $y = \dfrac{x^n}{n} + \dfrac{1}{Ax^p}$, how must A and p be related to n?

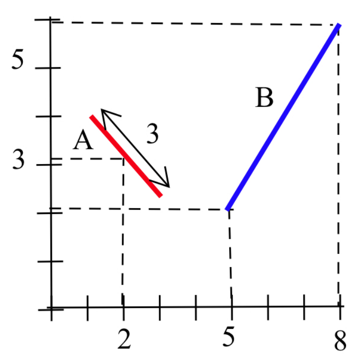
29. Use the formulas for A and p from the previous problem with $n = \frac{3}{2}$ and find a new function $y = \dfrac{2}{3}x^{\frac{3}{2}} + \dfrac{1}{Ax^p}$ so that $1 + \left[\frac{dy}{dx}\right]^2$ is a perfect square.

30. Find the surface area when each line segment in the figure below is rotated about the (a) x-axis and (b) y-axis.



31. Find the surface area when each line segment in the figure above is rotated about the line (a) $y = 1$ and (b) $x = -2$.

32. Find the surface area when each line segment in the figure below is rotated about the line (a) $y = 1$ and (b) $x = -2$.



33. Find the surface area when each line segment in the figure above is rotated about the (a) x-axis and (b) y-axis.

34. A line segment of length 2 with midpoint $(2, 5)$ makes an angle of θ with the horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the y-axis? Explain your reasoning.

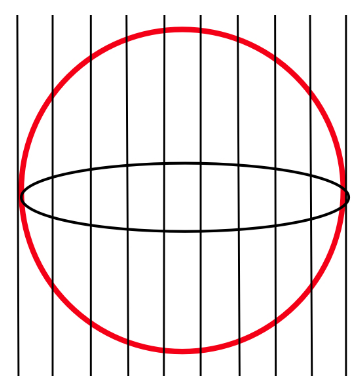
35. A line segment of length 2 with one end at $(2, 5)$ makes an angle of θ with the horizontal. What value of θ will result in the largest surface area when the line segment is rotated about the x-axis? Explain your reasoning.

In Problems 36–43, when the given curve is rotated about the given axis, represent the area of the resulting surface as a definite integral, then evaluate that integral using technology.

36. $y = x^3$ for $0 \le x \le 2$ about the y-axis
37. $y = 2x^3$ for $0 \le x \le 1$ about the y-axis
38. $y = x^2$ for $0 \le x \le 2$ about the x-axis
39. $y = 2x^2$ for $0 \le x \le 1$ about the x-axis
40. $y = \sin(x)$ for $0 \le x \le \pi$ about the x-axis
41. $y = x^3$ for $0 \le x \le 2$ about the x-axis
42. $y = \sin(x)$ for $0 \le x \le \dfrac{\pi}{2}$ about the y-axis
43. $y = x^2$ for $0 \le x \le 2$ about the y-axis

44. Find the area of the surface formed when the graph of $y = \sqrt{4 - x^2}$ is rotated about the x-axis:
    (a) for $0 \le x \le 1$.
    (b) for $1 \le x \le 2$.
    (c) for $2 \le x \le 3$.

45. Show that if a thin hollow sphere is sliced into pieces by equally spaced parallel cuts (see below), then each piece has the same weight. (Hint: Does each piece have the same surface area?)



46. Interpret the result of the previous problem for an orange sliced by equally spaced parallel cuts.

47. A hemispherical cake with a uniformly thick layer of frosting is sliced with equally spaced parallel cuts. Does everyone get the same amount of cake? The same amount of frosting?

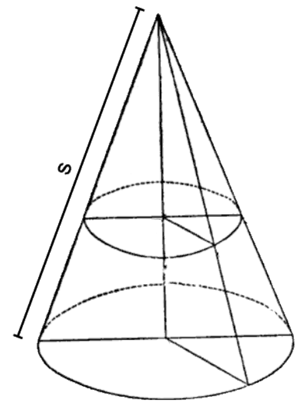
48. Devise a formula for the area of the surface generated by revolving the curve $x = g(y)$ for $c \le y \le d$ about the (a) x-axis and (b) y-axis.

49. Use the answer to the previous problem to find the area of the surface generated by revolving $x = e^y$ for $0 \le y \le 1$ about (a) the x-axis and (b) the y-axis.

50. Devise a formula for the area of the surface generated by revolving the curve given by parametric equations $x = x(t)$ and $y = y(t)$ for $\alpha \le t \le \beta$ about the (a) x-axis and (b) y-axis.

51. Use the answer to the previous problem to find the area of the surface generated by revolving the curve given by $x = \cos(t)$ and $y = \sin(t)$ for $0 \le t \le \frac{\pi}{2}$ about (a) the x-axis and (b) the y-axis.

52. The surface generated by revolving a line segment of length L about a line P (that does not intersect the line segment) is the **frustrum** of a cone: the surface that results from taking a larger cone of radius r_2 and removing a smaller cone of radius r_1 ("chopping off the top"). We know from geometry that the surface area of a cone is πrs where r is the radius of the cone and s is the **slant height**:



(a) If s_1 is the slant height of the smaller cone that is removed from the bigger cone, show that:

$$s_1 + L = \frac{r_2 L}{r_2 - r_1}$$

(Hint: Use similar triangles.)

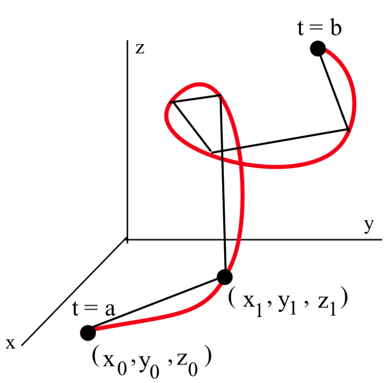
(b) Show that the surface area of the frustrum is:

$$\pi r_2 (s_1 + L) - \pi r_1 s_1$$

(c) Show that this quantity equals:

$$\pi (r_1 + r_2) L$$

(d) Show that this last quantity is the product of the distance traveled by the midpoint and the length of the line segment.



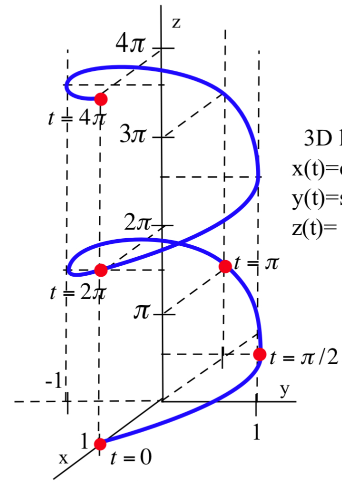
3-D Arclength

If a 3-dimensional curve $\mathcal{C}$ (see margin) is given parametrically by $x = x(t)$, $y = y(t)$ and $z = z(t)$ for $\alpha \le t \le \beta$, then we can easily extend the arclength formula to three dimensions:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The remaining problems in this section use this formula to provide you with a preview of calculus in higher dimensions.

53. Find the length of the helix (see figure) given by $x = \cos(t)$, $y = \sin(t)$, $z = t$ for $0 \le t \le 4\pi$.
54. Find the length of the line segment given by $x = t$, $y = t$, $z = t$ for $0 \le t \le 1$.
55. Find the length of the curve given by $x = t$, $y = t^2$, $z = t^3$ for $0 \le t \le 1$.
56. Find the length of the "stretched helix" given by $x = \cos(t)$, $y = \sin(t)$, $z = t^2$ for $0 \le t \le 2\pi$.
57. Find the length of the curve given by $x = 3\cos(t)$, $y = 2\sin(t)$, $z = \sin(7t)$ for $0 \le t \le 2\pi$.



5.3 Practice Answers

1. At least $2 + \sqrt{2} + \sqrt{13} + 1 + \sqrt{2} \approx 9.43$ miles.
2. $L \approx \sqrt{2} + \sqrt{10} + \sqrt{26} <$ actual length
3. $\displaystyle\int_1^4 \sqrt{1 + [2x]^2}\,dx = \int_1^4 \sqrt{1 + 4x^2}\,dx \approx 15.34$
4. $\displaystyle\int_0^{2\pi} \sqrt{1 + [\cos(x)]^2}\,dx \approx 7.64$
5. Here $g(y) = \sqrt{y} = y^{\frac{1}{2}} \Rightarrow g'(y) = \frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}$ so the arclength is:

$$\int_1^{16} \sqrt{1 + \left[\frac{1}{2\sqrt{y}}\right]^2}\,dy = \int_1^{16} \sqrt{1 + \frac{1}{4y}}\,dy \approx 15.34$$

This answer is the same as the answer to Practice 3. Should that surprise you?

6. Here $x'(t) = -\sin(t)$ and $y'(t) = \cos(t)$ so the arclength is:

$$\int_0^{2\pi} \sqrt{[-\sin(t)]^2 + [\cos(t)]^2}\,dt = \int_0^{2\pi} \sqrt{1}\,dt = 2\pi$$

The curve in question is a circle of radius 1. Does the answer from the integral formula agree with the answer you can obtain using simple geometry?

7. Here $x'(t) = 3$ and $y'(t) = 4$ so the arclength is:

$$\int_1^3 \sqrt{[3]^2 + [4]^2}\,dt = \int_1^3 5\,dt = 10$$

The "curve" is a line segment from $(4, 4)$ to $(10, 12)$. Does the answer from the integral formula agree with the answer you can obtain using simple geometry?

8. The surface area of the horizontal segment revolved about x-axis is $2\pi(1)(2) = 4\pi \approx 12.57$ while the surface area of other segment revolved about the x-axis is $2\pi(2)(\sqrt{8}) \approx 35.54$, so the total surface area is approximately $12.57 + 35.54 = 48.11$ square units.

   The surface area of the horizontal segment revolved about y-axis is $2\pi(3)(2) = 12\pi \approx 37.70$ while the surface area of the other segment revolved about the y-axis is $2\pi(5)(\sqrt{8}) \approx 88.86$, so the total surface area is approximately $37.70 + 88.86 = 126.56$ square units.