10.2 Power Series: $\sum_{k=0}^{\infty} a_k(x-c)^k$

The preceding section examined power series of the form $\sum_{k=0}^{\infty} a_k x^k$, which were guaranteed to converge at $x = 0$. As we observed, such a power series may have converged for all values of x, or for some smaller interval centered at $x = 0$. When we apply power series to approximate functions and solve problems throughout the rest of this chapter, we may need our power series to converge near some number other than $x = 0$. We can accomplish this by shifting a power series of the form $\sum_{k=0}^{\infty} a_k x^k$ to get a power series of the form $\sum_{k=0}^{\infty} a_k(x-c)^k$.

Definition:
A **power series** is an expression of the form:

$$\sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

where a_0, a_1, a_2, a_3,... are constants, called the **coefficients** of the series, and x is a variable.

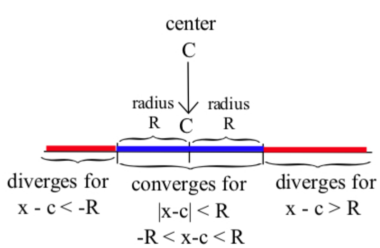
For $k = 0$ we again use the convention for power series that $(x-c)^0 = 1$ even when $x = c$.

When $x = c$, this series yields:

$$\sum_{k=0}^{\infty} a_k(c-c)^k = a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 + a_3 \cdot 0^3 + \cdots = a_0$$

so it is guaranteed to converge at $x = c$, no matter the values of a_k.

Furthermore, $x = c$ is always the center of the interval of convergence for this power series (just as $x = 0$ was the center of the intervals of convergence for the series we studied in the preceding section). The radius of convergence is—like it was in Section 10.1—half the length of the interval of convergence (see margin figure). If a power series centered at $x = c$ converges for some value of x, then the series converges for all values of x closer to $x = c$; if a power series centered at $x = c$ diverges for another value of x, then the series diverges for all values of x farther from $x = c$.

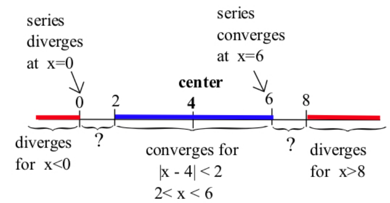


Example 1. If you know $\sum_{k=0}^{\infty} a_k(x-4)^k$ converges at $x = 6$ and diverges at $x = 0$, what can you conclude ("converge" or "diverge" or "not enough information") about the series when $x = 3, 9, -1, 2$ and 7?

Solution. We know the power series converges at $x = 6$, so we can conclude that the series converges for all values of x closer to 4 than $|6-4| = 2$ units: in particular, the series converges at $x = 3$.

We know the power series diverges at $x = 0$, so we can conclude that the series diverges for all values of x farther from 4 than $|0 - 4| = 4$ units: in particular, the series diverges at $x = 9$ and $x = -1$.

The remaining values of x (2 and 7) do not satisfy $|x - 4| < 2$ or $|x - 4| > 4$, so the series may converge or may diverge for those values of x. The margin figure shows the regions where convergence of this power series is guaranteed, where divergence is guaranteed, and where we lack enough information to make a determination. ◀



Practice 1. If you know that the power series $\sum_{k=0}^{\infty} a_k(x+5)^k$ converges at $x = -1$ and diverges at $x = 1$, what can you conclude about the series when $x = -2, -9, 0, -11$ and 3?

The Ratio Test remains our primary tool for finding an interval of convergence of a power series, even if the power series is centered at $x = c$ rather than at $x = 0$.

Example 2. Find the interval and radius of convergence of $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k \cdot 2^k}$.

Solution. Applying the Ratio Test:

$$\left| \frac{\frac{(x-5)^{k+1}}{(k+1)2^{k+1}}}{\frac{(x-5)^k}{k \cdot 2^k}} \right| = \left| \frac{(x-5)^{k+1}}{(k+1)2^{k+1}} \cdot \frac{k \cdot 2^k}{(x-5)^k} \right| = \left(\frac{k}{k+1} \right) \cdot \frac{|x-5|}{2} \longrightarrow \frac{|x-5|}{2}$$

The Ratio Test guarantees absolute convergence when:

$$\frac{|x-5|}{2} < 1 \quad \Rightarrow \quad |x-5| < 2 \quad \Rightarrow \quad -2 < x - 5 < 2 \quad \Rightarrow 3 < x < 7$$

and says that the power series diverges if $x < 3$ or $x > 7$. We still need to check the endpoints of this interval of convergence: $x = 3$ and $x = 7$.

When $x = 3$, the power series becomes:

$$\sum_{k=1}^{\infty} \frac{(3-5)^k}{k \cdot 2^k} = \sum_{k=1}^{\infty} \frac{(-2)^k}{k \cdot 2^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

(the alternating harmonic series), which converges conditionally. When $x = 7$, the power series becomes:

$$\sum_{k=1}^{\infty} \frac{(7-5)^k}{k \cdot 2^k} = \sum_{k=1}^{\infty} \frac{(2)^k}{k \cdot 2^k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

(the harmonic series), which diverges. The interval of convergence is therefore $[3, 7)$ and the radius of convergence is $R = \frac{1}{2}(7-3) = 2$. ◀

This power series converges *absolutely* on the interval $(3, 7)$.

Practice 2. Find the interval and radius of convergence of $\sum_{k=1}^{\infty} \frac{k \cdot (x-3)^k}{5^k}$.

10.2 Problems

In Problems 1–15, determine the interval of convergence and radius of convergence for the given power series, then graph the interval of convergence on a number line.

1. $\sum_{k=0}^{\infty} (x+2)^k$
2. $\sum_{k=0}^{\infty} (x-3)^k$
3. $\sum_{k=1}^{\infty} (x+5)^k$
4. $\sum_{k=1}^{\infty} \frac{(x+3)^k}{k}$
5. $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k}$
6. $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$
7. $\sum_{k=1}^{\infty} \frac{(x-7)^{2k+1}}{k^2}$
8. $\sum_{k=1}^{\infty} \frac{(x+1)^{2k}}{k^3}$
9. $\sum_{k=0}^{\infty} (2x-6)^k$
10. $\sum_{k=0}^{\infty} (3x+1)^k$
11. $\sum_{k=0}^{\infty} \frac{(x-5)^k}{k!}$
12. $\sum_{k=0}^{\infty} k! \cdot (x+2)^k$
13. $\sum_{k=3}^{\infty} k! \cdot (x-7)^k$
14. $\sum_{k=3}^{\infty} (x-7)^{2k}$
15. Your friend claims the interval of convergence for a power series of the form $\sum_{k=0}^{\infty} a_k(x-4)^k$ is the interval $(1,9)$. Without checking his work, how can you be certain that your friend is wrong?
16. Determine which of the following intervals *could* be the interval of convergence for a power series of the form $\sum_{k=0}^{\infty} a_k(x-4)^k$: $(2,6)$, $(0,4)$, $\{0\}$, $[1,7]$, $(-1,9]$, $\{4\}$, $[3,5)$, $[-4,4)$, $\{3\}$.
17. Determine which of the following intervals *could* be the interval of convergence for a power series of the form $\sum_{k=0}^{\infty} a_k(x-7)^k$: $(3,10)$, $(5,9)$, $\{0\}$, $[1,13]$, $(-1,15]$, $\{4\}$, $[3,11)$, $[0,14)$, $\{7\}$.
18. Fill in each blank with a number so the resulting interval *could* be the interval of convergence for a power series of the form $\sum_{k=0}^{\infty} a_k(x-3)^k$: $(0,\_\_)$, $(\_\_,7)$, $[0,\_\_]$, $(\_\_,15]$, $[\_\_,11)$, $[0,\_\_)$, $\{\_\_\}$.
19. Fill in each blank with a number so the resulting interval *could* be the interval of convergence for a power series of the form $\sum_{k=0}^{\infty} a_k(x-1)^k$: $(0,\_\_)$, $(\_\_,7)$, $[0,\_\_]$, $(\_\_,5]$, $[\_\_,11)$, $[0,\_\_)$, $\{\_\_\}$.
20. Fill in each blank with a number so the resulting interval *could* be the interval of convergence for a power series of the form $\sum_{k=0}^{\infty} a_k(x+1)^k$: $(-2,\_\_)$, $(\_\_,7)$, $[-3,\_\_]$, $(\_\_,5]$, $[\_\_,11)$, $[-4,\_\_)$, $\{\_\_\}$.

In Problems 21–24, given the interval of convergence for a power series, find its radius of convergence.

21. $(0,6)$ 22. $[0,8)$ 23. $(2,8]$ 24. $[3,7]$

In Problems 25–28, use the patterns you noticed in earlier problems and Examples to build a power series with the given interval of convergence. (There are many possible correct answers—find one.)

25. $(0,6)$ 26. $[0,8)$ 27. $(2,8]$ 28. $[3,7]$

In Problems 29–34, find the interval of convergence for each series. Then, for x in the interval of convergence, find the sum of the series as a function of x. (Hint: You already know how to find the sum of a geometric series.)

29. $\sum_{k=0}^{\infty} (x-3)^k$
30. $\sum_{k=0}^{\infty} \left(\frac{x-6}{2}\right)^k$
31. $\sum_{k=0}^{\infty} \left(\frac{x-6}{5}\right)^k$
32. $\sum_{k=0}^{\infty} \left(\frac{x}{3}\right)^{2k}$
33. $\sum_{k=0}^{\infty} \left(\frac{1}{2}\sin(x)\right)^k$
34. $\sum_{k=0}^{\infty} \left(\frac{1}{2}\cos(x)\right)^k$

In 35–42, the letters a and b represent positive constants. Find the interval of convergence for each series.

35. $\displaystyle\sum_{k=1}^{\infty} (x-a)^k$

36. $\displaystyle\sum_{k=1}^{\infty} (x+b)^k$

37. $\displaystyle\sum_{k=1}^{\infty} \frac{(x-a)^k}{k}$

38. $\displaystyle\sum_{k=1}^{\infty} \frac{(x-a)^k}{k^2}$

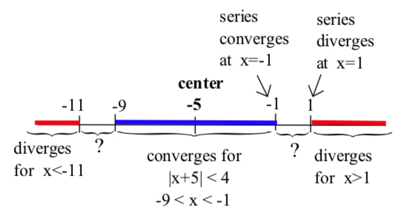
39. $\displaystyle\sum_{k=1}^{\infty} (ax)^k$

40. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{x}{a}\right)^k$

41. $\displaystyle\sum_{k=1}^{\infty} (ax-b)^k$

42. $\displaystyle\sum_{k=1}^{\infty} (ax+b)^k$

10.2 Practice Answers



1. The center of the power series is at $x=-5$ and the series converges at $x=-1$, so it must converge at all points less than $|-1-(-5)|=4$ units from $x=-5$, in particular at $x=-2$. The series diverges at $x=1$, so it must diverge at all points more than $|1-(-5)|=6$ units from $x=-5$, in particular at $x=3$. The series may converge or diverge at the other points. See the margin figure for a graph of the regions of known convergence and known divergence.

2. Applying the Ratio Test:

$$\left|\frac{\frac{(k+1)\cdot(x-3)^{k+1}}{5^{k+1}}}{\frac{k\cdot(x-3)^k}{5^k}}\right| = \left|\frac{(k+1)\cdot(x-3)^{k+1}}{5^{k+1}}\cdot\frac{5^k}{k\cdot(x-3)^k}\right| = \left(\frac{k+1}{k}\right)\cdot\frac{|x-3|}{5} \to \frac{|x-3|}{5}$$

The Ratio Test tells us that the power series converges when:

$$\frac{|x-3|}{5} < 1 \;\Rightarrow\; |x-3| < 5 \;\Rightarrow\; -5 < x-3 < 5 \;\Rightarrow -2 < x < 8$$

and that the power series diverges for $x<-2$ and $x>8$. We need to check the endpoints of this interval separately. When $x=-2$ the power series becomes:

$$\sum_{k=1}^{\infty} \frac{k\cdot(-2-3)^k}{5^k} = \sum_{k=1}^{\infty} (-1)^k\cdot k$$

which diverges (by the Test for Divergence). When $x=8$, the power series becomes:

$$\sum_{k=1}^{\infty} \frac{k\cdot(8-3)^k}{5^k} = \sum_{k=1}^{\infty} k$$

which also diverges (by the Test for Divergence). The interval of convergence is therefore $(-2,8)$ and the radius of convergence is $R=\frac{1}{2}\cdot 10 = 5$ (half the length of the interval of convergence).